ITERATIONS OF A KIND OF EXPONENTIALS

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1. Introduction

We shall study a sequence of numbers defined recursively. Let ln denote the principal branch of the natural logarithm, i.e., $\ln(re^{i\theta}) = \ln r + i\phi$, $r > 0$, with $\phi \equiv \theta \pmod{2\pi}$, $-\pi < \phi \le \pi$. We put $0 \,\square\, z := z$, $1 \,\square\, z := z^z (:= e^{z \ln z})$ and

$$(1) \qquad (n+1) \,\square\, z = (n \,\square\, z)^{(n \,\square\, z)}, \quad n = 0, 1, 2, \ldots$$

$$(n \,\square\, 1 = 1,\ n \,\square\, (-1) = -1,\ 1 \,\square\, i = e^{-\pi/2}).$$

We consider, in fact, a more general operation defined by

$$\alpha_0(a, b) := b,\ \alpha_1(a, b) := b^{b^a}$$

and

$$(2) \qquad \alpha_{n+1}(a, b) := \alpha_n(a, b)^{\alpha_n^a(a, b)}, \quad n = 0, 1, 2, \ldots$$

$$\left(\alpha_n(1, z) = n \,\square\, z,\ n \,\square\, i = \alpha_{n-1}^{-\pi/2}\left(-\frac{\pi}{2}, e\right)\right).$$

By mathematical induction, we obtain the

Proposition: The following algebraic relations hold for all $n, m \in \mathbb{N}$ and $a, b, c, z \in \mathbb{C}$:

a) $\alpha_{n+m}(a, b) = \alpha_n(a, \alpha_m(a, b))$ [in particular $(n + m) \,\square\, z = n \,\square\, (m \,\square\, z)$].

b) $\alpha_n(a, b^c) = \alpha_n^c(ac, b)$ [in particular $n \,\square\, z^c = \alpha_n^c(c, z)$ and $\alpha_n(a, b^a) = \alpha_n^a(a^2, b)$].

c) $\alpha_n(a, b) = b^{\prod_{k=0}^{n-1} \alpha_k^a(a, b)}$ $\left(\text{in particular } n \,\square\, z = z^{\prod_{k=0}^{n-1} k \,\square\, z}\right)$.

It will be proved in the paper that

$$(3) \qquad \lim_{n \to \infty} n \,\square\, e^{z/n} = 1, \quad |z| < \frac{1}{e},\ z \in \mathbb{C}.$$

Moreover, the inverse function of ψ, $\psi(z) := n \,\square\, z$, is explicitly calculated for $|z| \le 1/e$, and we examine the possibility to extend the definition of $\zeta \,\square\, z$ to complex values of ζ.

2. The Evaluation of a Limit

The evaluation (3) is an immediate consequence of

Theorem 1: For all positive integers n and complex numbers z such that $|z| < 1/e$, we have

$$(4) \qquad \left|\ln(n \,\square\, e^{z/n})\right| \le \frac{1}{n} \sum_{\nu=1}^{\infty} \frac{\nu^\nu}{\nu!} |z|^\nu.$$

The following lemma is useful to prove (4) (in [2], see formula (15) and section 4.1).

Lemma 1: Let $f_0^{(4)} := f$, $f_1^{(4)}(z) := \exp\left(\dfrac{zf'(z)}{f(z)}\right)$ and $f_{m+1}^{(4)} := (f_m^{(4)})_1^{(4)}$, $m = 1, 2, 3, \ldots$. We have

(5) $(f(z^z))_m^{(4)} = \prod_{k=1}^{m} \prod_{j=0}^{k} f_k^{(4)}(z^z)^{\omega(m,k,j)z^k(\ln z)^j}$

where

$$j!\omega(m, k, j) = \binom{m}{k-j}\sum_{s=0}^{j}(-1)^s\binom{j}{s}(k-s)^{m-k+j}, \quad 0 \le j \le k, \quad 1 \le k \le m.$$

In particular,

(6) $(f(z^z))_m^{(4)}(z = 1) = \prod_{k=1}^{m} f_{k(1)}^{(4)\binom{m}{k}k^{m-k}}$.

Proof of Theorem 1: We apply (6) recursively to

$$f(\zeta) = (n-1) \square \zeta, \ (n-2) \square \zeta, \ \ldots, \ 1 \square \zeta.$$

Using $n \square \zeta = (n-1) \square \zeta^\zeta$, we get

(7) $(n \square \zeta)^{(4)}(\zeta = 1) = \prod_{k=1}^{m}((n-1) \square \zeta)_k^{(4)^{k^{m-k}\binom{m}{k}}}(\zeta = 1)$.

At the r^{th} step, we obtain ($k_0 := m$):

(8) $(n \square \zeta)_m^{(4)}(\zeta = 1)$

$$= \prod_{k_1=1}^{m} \cdots \prod_{k_r=1}^{k_{r-1}} \left((n-r) \square \zeta\right)_{k_r}^{(4)}(\zeta = 1)^{k^{m-k_1}\binom{m}{k_1}\cdots k_r^{k_{r-1}-k_r}\binom{k_{r-1}}{k_r}},$$

whence, since $(1 \square \zeta)_\nu^{(4)}(\zeta = 1) = e^\nu$, $\nu = 0, 1, 2, \ldots$,

(9) $(n \square \zeta)_m^{(4)}(\zeta = 1)$

$$= \prod_{k_1=1}^{m} \cdots \prod_{k_{n-1}=1}^{k_{n-2}} \exp\left(k_{n-1} \cdot k_{n-1}^{k_{n-2}-k_{n-1}}\binom{k_{n-2}}{k_{n-1}} \cdots k_1^{m-k_1} \cdot \binom{m}{k_1}\right)$$

It follows from (9) that

(10) $\exp(m \cdot n^{m-1}) \le (n \square \zeta)_m^{(4)}(\zeta = 1)$

$$\le \exp\left(m^{m-1} \cdot \sum_{k_1=1}^{m} \cdots \sum_{k_{n-1}=1}^{k_{n-2}} k_{n-1} \cdot \binom{k_{n-2}}{k_{n-1}} \cdots \binom{m}{k_1}\right)$$

$$= \exp(m^m \cdot n^{m-1}) \quad \left(\text{we use } \sum_{j=1}^{N} j\binom{N}{j}x^{j-1} = N(1+x)^{N-1}\right).$$

Thus, the series

$$\sum_{m=1}^{\infty} \frac{\ln((n \square \zeta)_m^{(4)}(\zeta = 1))Z^m}{m!} \text{ converges for } |Z| < \frac{1}{ne} \text{ and}$$

(11) $\left|\sum_{m=1}^{\infty} \frac{\ln((n \square \zeta)_m^{(4)}(\zeta = 1))}{m!n^m} z^m\right| \le \frac{1}{n}\sum_{m=1}^{\infty}\frac{m^m}{m!}|z|^m, \quad |z| < \frac{1}{e}.$

Let us observe that, in general,

(12) $F_m^{(4)}(z_0) = \exp\left(\frac{\partial^m}{\partial \omega^m} \ln F(z_0 e^w)\Big|_{w=0}\right).$

In our case

$$\ln(n \square \zeta)_m^{(4)}(\zeta = 1) = \frac{\partial^m}{\partial w^m}\ln(n \square e^w)\Big|_{w=0}$$

so that the MacLaurin expansion of $\ln(n \square e^{z/n})$, namely,

(13) $\ln(n \square e^{z/n}) = \sum_{m=1}^{\infty} \frac{\ln(n \square \zeta)_m^{(4)}(\zeta = 1)}{m!n^m} z^m,$

is valid for $|z| < 1/e$ in view of (11). This completes the proof of Theorem 1, since (4) follows from (11) and (13). □

3. The Inverse Function

If $\zeta = n \,\square\, z$, $n = 1, 2, 3, \ldots$, then we write $z = (-n) \,\square\, \zeta$ in a domain where the inverse function is defined (this is essentially what is called "partial inverse" in [3]). The inverse function is defined in such a way that

$$(14) \qquad (n + m) \,\square\, z = n \,\square\, (m \,\square\, z), \quad n, m \in \mathbb{Z}.$$

To prove the next theorem, we need the following lemma.

Lemma 2: For all complex numbers $A_1, A_2, \ldots, A_m$, we have

$$(15) \qquad \sum_{\pi(m,r)} \frac{r!}{k_1! \ldots k_m!} \prod_{j=1}^{m} A_j^{k_j} = \sum_{\substack{\nu_1 + \cdots + \nu_r = m \\ \nu_\ell \geq 1}} \prod_{\ell=1}^{r} A_{\nu_\ell}, \quad 1 \leq r \leq m.$$

Here and in what follows, $\pi(m, r)$ means that the summation is extended over the numbers $k_1, \ldots, k_m$ such that

$$k_1 + 2k_2 + \cdots + mk_m = m, \; k_1 + k_2 + \cdots + k_m = r,$$

with $k_j \geq 0$, $1 \leq j \leq m$.

Proof: Let

$$f(z) := \sum_{m=1}^{\infty} B_m z^m, \quad g(z) := \sum_{m=1}^{\infty} A_m z^m$$

be two analytic functions in a neighborhood of $z = 0$ such that $f(0) = g(0) = 0$. We have

$$f(g(z)) = \sum_{m=1}^{\infty} B_m (g(z))^m = \sum_{m=1}^{\infty} \sum_{\nu_1=1}^{\infty} \cdots \sum_{\nu_m=1}^{\infty} B_m A_{\nu_1} \ldots A_{\nu_m} z^{\nu_1 + \cdots + \nu_m}$$

$$= \sum_{m=1}^{\infty} \sum_{r=m}^{\infty} \sum_{\substack{\nu_1 + \cdots + \nu_m = r \\ \nu_\ell \geq 1}} B_m A_{\nu_1} \ldots A_{\nu_m} z^{\nu_1 + \cdots + \nu_m},$$

whence

$$(16) \qquad f(g(z)) = \sum_{m=1}^{\infty} \sum_{r=1}^{m} \sum_{\substack{\nu_1 + \cdots + \nu_r = m \\ \nu_\ell \geq 1}} B_r \prod_{\ell=1}^{r} A_{\nu_\ell} \cdot z^m$$

i.e.,

$$(17) \qquad \frac{(f(g(z)))^{(m)}(z = 0)}{m!} = \sum_{r=1}^{m} \sum_{\substack{\nu_1 + \cdots + \nu_r = m \\ \nu_\ell \geq 1}} B_r \prod_{\ell=1}^{r} A_{\nu_\ell}.$$

On the other hand, we compute $(f(g(z)))^{(m)}$ using the Faa di Bruno formula [5, p. 177], namely,

$$(18) \qquad (f(g(z)))^{(m)} = \sum_{r=1}^{m} \sum_{\pi(m,r)} \frac{m!}{k_1! \ldots k_m!} \prod_{j=1}^{m} \left(\frac{g^{(j)}(z)}{j!}\right)^{k_j} \cdot f^{(r)}(g(z)).$$

It gives us

$$(19) \qquad \frac{(f(g(z)))^{(m)}(z = 0)}{m!} = \sum_{r=1}^{m} \sum_{\pi(m,r)} \frac{r!}{k_1! \ldots k_m!} B_r \prod_{j=1}^{m} A_j^{k_j},$$

and the result follows by comparison of (17) and (19).

Remark: Formula (15) gives a variant of (18):

$$(20) \qquad \frac{(f(g(z)))^{(m)}}{m!} = \sum_{r=1}^{m} \sum_{\substack{\nu_1 + \cdots + \nu_r = m \\ \nu_\ell \geq 1}} \prod_{\ell=1}^{r} \left(\frac{g^{(\nu_\ell)}(z)}{\nu_\ell!}\right) \cdot \frac{f^{(r)}(g(z))}{r!}.$$

We shall also need

Lemma 3 [2, p. 238]: For all analytic functions $\phi(z)$, we have

$$\sum_{\pi(m,r)} \frac{m!}{k_1! \ldots k_m!} \prod_{j=1}^{m} \left(\frac{(\phi^j(z))^{(j-1)}}{j!}\right)^{k_j} = \binom{m-1}{r-1} (\phi^m(z))^{(m-r)}, \quad 1 \le r \le m. \tag{21}$$

A representation of $(-1) \square y$ is obtainable from the results of [3] (an interesting list of references is given in that paper). It is proved that the function

$$x = h(z) = z^{z^{z^{\cdot^{\cdot^{\cdot}}}}}$$

converges when $e^{-e} \le z \le e^{1/e}$; moreover,

$$g(h(z)) = z \quad \text{and} \quad h(g(x)) = x, \quad e^{-1} \le x \le e,$$

where

$$g(x) = x^{1/x}.$$

But

$$\frac{1}{g\left(\frac{1}{x}\right)} = 1 \square x =: y,$$

whence

$$g\left(\frac{1}{x}\right) = \frac{1}{y}, \quad \frac{1}{x} = h\left(\frac{1}{y}\right) \quad \text{for } e^{-1/e} \le y \le e^e,$$

i.e.,

$$x = \frac{1}{h\left(\frac{1}{y}\right)} = (-1) \square y,$$

whence

$$(-1) \square y = y^{y^{-y^{-y^{-y^{\cdot^{\cdot^{\cdot}}}}}}}, \quad e^{-1/e} \le y \le e^e. \tag{22}$$

Replacing y by $(-1) \square y$ gives a similar representation for $(-2) \square y$, and so on. We give here another kind of representation for $(-m) \square z$, $m = 1, 2, 3, \ldots$.

Theorem 2: For all positive integers m and complex numbers z such that

$$|\ln z| \le \frac{1}{me},$$

we have

$$(-m) \square z = \prod_{\nu=1}^{\infty} \prod_{\nu_1=1}^{\nu} \cdots \prod_{\nu_{m-1}=1}^{\nu_{m-2}} \exp\left(\frac{(-1)^{\nu-1}}{\nu!} \cdot \binom{\nu-1}{\nu_1-1} \cdots \binom{\nu_{m-2}-1}{\nu_{m-1}-1} \cdot \nu^{\nu-\nu_1} \cdots \nu_{m-2}^{\nu_{m-2}-\nu_{m-1}} \cdot \nu_{m-1}^{\nu_{m-1}-1} \cdot (\ln z)^{\nu}\right). \tag{23}$$

Proof: According to the Lagrange expansion theorem, the root z of the equation $z \ln z = \ln \zeta$ which tends to 1 with ζ is given by

$$\ln z = \sum_{\nu=1}^{\infty} (-1)^{\nu-1} \frac{\nu^{\nu-1}}{\nu!} (\ln \zeta)^{\nu}, \quad |\ln \zeta| \le \frac{1}{e}.$$

Since $z \ln z = \ln \zeta$ implies $\zeta = z^z = 1 \square z$, we obtain

$$\ln((-1) \square \zeta) = \sum_{\nu=1}^{\infty} (-1)^{\nu-1} \frac{\nu^{\nu-1}}{\nu!} (\ln \zeta)^{\nu}, \quad |\ln \zeta| \le \frac{1}{e}, \tag{24}$$

which corresponds to (23) for $m = 1$.

Now we replace ζ by $(-1) \square \zeta$ in (24) to obtain

$$\ln((-2) \square \zeta) = \sum_{\nu=1}^{\infty} (-1)^{\nu-1} \frac{\nu^{\nu-1}}{\nu!} \sum_{k_1=1}^{\infty} \cdots \sum_{k_\nu=1}^{\infty} (-1)^{k_1+\cdots+k_\nu-\nu} \cdot \frac{k_1^{k_1-1} \cdots k_\nu^{k_\nu-1}}{k_1! \cdots k_\nu!} \cdot (\ln \zeta)^{k_1+\cdots+k_\nu}$$

$$= \sum_{\nu=1}^{\infty} \sum_{\mu=\nu}^{\infty} \sum_{\substack{k_1+\cdots+k_\nu=\mu \\ k_\ell \ge 1}} (-1)^{\mu-1} \frac{\nu^{\nu-1}}{\nu!} \prod_{\ell=1}^{\nu} \left(\frac{k_\ell^{k_\ell-1}}{k_\ell!}\right) \cdot (\ln \zeta)^{\mu},$$

i.e.,

$$\ln((-2) \square \zeta) = \sum_{\nu=1}^{\infty} \sum_{\mu=1}^{\nu} \sum_{\substack{k_1+\cdots+k_\mu=\nu \\ k_\ell \ge 1}} (-1)^{\nu-1} \frac{\mu^{\mu-1}}{\mu!} \prod_{\ell=1}^{\mu} \left(\frac{k_\ell^{k_\ell-1}}{k_\ell!}\right) \cdot (\ln \zeta)^{\nu}. \tag{25}$$

The identity (15) with $A_j = \dfrac{j^{j-1}}{j!}$ gives

$$\sum_{\substack{k_1+\cdots+k_\mu=\nu \\ k_\ell \ge 1}} \prod_{\ell=1}^{\mu} \left(\frac{k_\ell^{k_\ell-1}}{k_\ell!}\right) = \sum_{\pi(\nu,\mu)} \frac{\mu!}{k_1! \cdots k_\nu!} \prod_{j=1}^{\nu} \left(\frac{j^{j-1}}{j!}\right)^{k_j}, \tag{26}$$

while (21) [with $\phi(z) = e^z$] gives

$$\sum_{\pi(\nu,\mu)} \frac{\nu!}{k_1! \cdots k_\nu!} \prod_{j=1}^{\nu} \left(\frac{j^{j-1}}{j!}\right)^{k_j} = \binom{\nu-1}{\mu-1} \nu^{\nu-\mu}, \quad 1 \le \mu \le \nu. \tag{27}$$

We obtain

$$\sum_{\substack{k_1+\cdots+k_\mu=\nu \\ k_\ell \ge 1}} \prod_{\ell=1}^{\mu} \left(\frac{k_\ell^{k_\ell-1}}{k_\ell!}\right) = \frac{\mu!}{\nu!} \binom{\nu-1}{\mu-1} \nu^{\nu-\mu}, \quad 1 \le \mu \le \nu, \tag{28}$$

and it follows from (25) that

$$\ln((-2) \square \zeta) = \sum_{\nu=1}^{\infty} \sum_{\mu=1}^{\nu} \frac{(-1)^{\nu-1}}{\nu!} \nu^{\nu-\mu} \mu^{\mu-1} \binom{\nu-1}{\mu-1} (\ln \zeta)^{\nu}. \tag{29}$$

It is readily seen that the coefficients in the summation over ν of (29) are bounded by

$$\frac{\nu^{\nu-1}}{2\nu!} (2|\ln \zeta|)^{\nu},$$

so that (29) is valid for $|\ln \zeta| \le 1/2e$.

The proof is easily completed by mathematical induction. We write

$$\ln((-(m+1)) \square \zeta) = \ln((-m) \square ((-1) \square \zeta)),$$

substitute z to $(-1) \square \zeta$ in (23), and use (28) to simplify the coefficients. The estimation

$$|(-m) \square \zeta| \le \exp\left(\frac{1}{m} \sum_{\nu=1}^{\infty} \frac{\nu^{\nu-1}}{\nu!} |m \ln \zeta|^{\nu}\right) \le e^{1/m} \tag{30}$$

holds for $|\ln \zeta| \le 1/me$. □

Remark: It follows from the proof of Theorem 2 that

$$\lim_{m \to \infty} (-m) \square \zeta^{1/m} = 1, \quad |\ln \zeta| \le \frac{1}{e}. \tag{31}$$

4. Extension of the Definition

In this section, we consider the possibility to define $\zeta \square z$ for complex values of ζ. We give only partial results, but it is interesting to observe that it seems quite possible to extend $\zeta \square z$ to a bianalytic function of z, ζ. All along the process, the relation

(32) $$(\zeta_1 + \zeta_2) \square z = \zeta_1 \square (\zeta_2 \square z)$$

should remain valid in some domains of the complex plane.

4.1 Extension to Rational Numbers

First, we try to see how $\frac{1}{2} \square z$ can be defined. Let us consider a more general question. Given $z_0 \in \mathbb{C}$ and

$$g(z) := \sum_{k=0}^{\infty} a_k (z - z_0)^k, \quad a_0 := z_0,$$

analytic in a neighborhood of z_0 (this fact will be abbreviated $z \circlearrowleft z_0$ in what follows), does there exist an analytic function

$$f(z) := \sum_{k=0}^{\infty} b_k (z - z_0)^k, \quad b_0 := z_0,$$

such that the functional equation

(33) $$f(f(z)) = g(z)$$

is valid for $z \circlearrowleft z_0$?

A solution is not always possible, as shown by the example

$$g(z) = z^2, \quad z_0 = 0.$$

An affirmative answer for $g(z) = z^z$, $z_0 = 1$, would imply that the solution $f(z) =: \frac{1}{2} \square z$ satisfies the relation

$$\frac{1}{2} \square \left(\frac{1}{2} \square z\right) = f(f(z)) = 1 \square z.$$

To solve the functional equation

(34) $$f(f(z)) = z^z, \quad f(1) = f'(1) = 1,$$

we seek a solution of the form

$$f(z) = 1 + \sum_{k=1}^{\infty} b_k (z - 1)^k.$$

Substituting z to $f(z)$, we obtain

$$z^z =: 1 + \sum_{k=1}^{\infty} a_k (z - 1)^k = 1 + \sum_{k=1}^{\infty} b_k (f(z) - 1)^k$$

$$= 1 + \sum_{k=1}^{\infty} \sum_{\ell=1}^{k} \sum_{\substack{\nu_1 + \cdots + \nu_\ell = k \\ \nu_j \geq 1}} b_\ell \prod_{j=1}^{\ell} b_{\nu_j} \cdot (z - 1)^k$$

(in the context of [2], it is not difficult to verify that $|a_k| \leq 1$ for all $k \in \mathbb{N}$). It is then readily seen that the aforesaid question can be answered in the affirmative if we find a practical way to solve the following two problems:

1. Express $b_1, b_2, \ldots, b_k$ in terms of $a_1, a_2, \ldots, a_k$ in the relations $a_1 = b_1 = 1$,

$$a_k = \sum_{r=1}^{k} \sum_{\substack{\nu_1 + \cdots + \nu_r = k \\ \nu_\ell \geq 1}} b_r \prod_{\ell=1}^{r} b_{\nu_\ell}, \quad k = 1, 2, 3, \ldots .$$

2. Show that the radius of convergence of $\sum_{k=1}^{\infty} b_k(z-1)^k$ is positive.

We assume in the remainder of the paper that the radius of convergence is positive in the case $g(z) = z^z$, $z_0 = 1$. Unfortunately, this fact is not proved but it seems very likely that it is ≥ 1.

We generalize one step further and ask for an analytic solution of

$$(35)\qquad f_q(z) = z^z, \quad f(1) = f'(1) = 1, \text{ where } f_q(z) = \underbrace{f(f(\cdots f(z)\cdots))}_{q \text{ times}}.$$

This leads us to define

$$(36)\qquad \frac{1}{q} \,\square\, z := f(z) := 1 + \sum_{k=1}^{\infty} b_k\left(\frac{1}{q}\right)(z-1)^k, \quad z \circlearrowleft 1,$$

for $q = 1, 2, 3, \ldots$ (the domain of validity should contain $|z - 1| < q/2$). It is then possible to define $p/q \,\square\, z$ for $p/q \in \mathbb{Q}_+$. Simply:

$$(37)\qquad \frac{p}{q} \,\square\, z := \underbrace{\frac{1}{q} \,\square\, \left(\frac{1}{q} \,\square\, \cdots \,\square\, \left(\frac{1}{q} \,\square\, z\right)\cdots\right)}_{p \text{ times}} =: 1 + \sum_{k=1}^{\infty} b_k(p, q)(z-1)^k, \quad z \circlearrowleft 1.$$

It appears that $b_k(p, q) = b_k(p/q)$. There is no problem defining $p/q \,\square\, z$ for $p/q \in \mathbb{Q}_-$. We construct $(-1)/q \,\square\, z$ by requiring

$$\frac{(-1)}{q} \,\square\, \left(\frac{1}{q} \,\square\, z\right) \equiv z$$

and we observe that (32) remains true for all rationals ζ_1, ζ_2. Here, we can write

$$(38)\qquad \frac{p}{q} \,\square\, z = z + \frac{p}{q}(z-1)^2 + \frac{p}{2q}\left(\frac{2p}{q} - 1\right)(z-1)^3 + \cdots, \quad z \circlearrowleft 1.$$

4.2 Extension to Complex Numbers

It is reasonable to expect that a passage to the limit can be justified in (38). This would permit us to define $t \,\square\, z$ for $t \in \mathbb{R}$ by

$$(39)\qquad t \,\square\, z := \lim_{j\to\infty} \sum_{k=0}^{\infty} b_k\left(\frac{p_j}{q_j}\right) \cdot (z-1)^k = \sum_{k=0}^{\infty} b_k(t) \cdot (z-1)^k, \quad z \circlearrowleft 1,$$

where p_j/q_j, $j = 1, 2, 3, \ldots$, is any sequence of rational numbers converging to t [note that the coefficients $b_k(t)$ are reals for real values of t).

Finally, (39) is extended to complex values of t by analytic continuation and (32) remains valid. We do not give details of our calculations, since the question concerning the radius of convergence is open. At the end of the process we obtain a representation of the form

$$(40)\qquad \zeta \,\square\, z = z + \zeta(z-1)^2 + \zeta\left(\zeta - \frac{1}{2}\right)(z-1)^3 + \cdots, \quad \zeta \circlearrowleft 0,\ z \circlearrowleft 1.$$

We can define $\alpha_\zeta(a, z)$ [see (2)] by requiring

$$\alpha_\zeta^a(a, z) = \zeta \,\square\, z^a.$$

5. Some Observations

5.1 Solution of a Functional Equation

We observe that the functional equation

$$(41)\qquad f_q(z) = z^N, \quad f(0) = 0, \quad N \in \mathbb{N}$$

can be solved.

Theorem 3: Let $N > 1$ be an integer. There exists an analytic solution, in a neighborhood of the origin, of the equation (41) if and only if $N = M^q$, $M \in \mathbb{N}$. The solution is unique up to a multiplicative constant which must be an $\left(\frac{N-1}{M-1}\right)^{\text{th}}$ root of unity.

Proof: If $N = M^q$, then a solution of (41) is

$$f(z) = cz^M, \quad c^{\frac{N-1}{M-1}} = 1.$$

We must prove that an analytic solution $f(z)$, $z \circlearrowright 0$, exists only in that case.

Equation (41) implies

$$f(z^N) = (f(z))^N, \quad f(0) = 0, \quad (N > 1). \tag{42}$$

Let us assume for a moment that the solutions of (42) are of the form

$$f(z) = cz^M, \quad c^N = c,$$

for some positive integer M. Substituting in (41), we find that

$$z^N = c^{1+M+\cdots+M^{q-1}} \cdot z^{M^q},$$

i.e., $N = M^q$ and $c^{\frac{N-1}{M-1}} = 1$. Hence, we need only to prove that all the analytic solutions of (42) are of the indicated form. Let

$$f(z) = \sum_{m=1}^{\infty} A_m z^m$$

be a solution of (42). We have

$$f(z^N) = \sum_{k=1}^{\infty} A_k z^{Nk} = (f(z))^N = \sum_{\nu_1=1}^{\infty} \cdots \sum_{\nu_N=1}^{\infty} A_{\nu_1} \cdots A_{\nu_N} \cdot z^{\nu_1+\cdots+\nu_N}$$

$$= \sum_{m=N}^{\infty} \sum_{\substack{\nu_1+\cdots+\nu_N=m \\ \nu_\ell \ge 1}} \prod_{\ell=1}^{N} A_{\nu_\ell} \cdot z^m,$$

whence

$$\sum_{\substack{\nu_1+\cdots+\nu_N=m \\ \nu_\ell \ge 1}} \prod_{\ell=1}^{N} A_{\nu_\ell} = \begin{cases} A_k & \text{if } m = kN,\ k \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases} \tag{43}$$

This relation, for $m = N$, gives $A_1^N = A_1$, i.e., $A_1 = 0$ or $A_1^{N-1} = 1$. The following reasoning is easily adapted to the case $A_1 \ne 0$ [we obtain the solution $f(z) = A_1 z$]. Let us suppose that $A_1 = 0$. Let $k_0 > 1$ be the first index for which $A_{k_0} \ne 0$. We prove by mathematical induction that $A_{k_0+\ell} = 0$, $\ell = 1, 2, 3, \ldots$ [this gives us the solution $f(z) = A_{k_0} z^{k_0}$, $A_{k_0}^N = A_{k_0}$).

First, we examine the relation (43) with $m = Nk_0 + 1$. If a ν_ℓ is less than k_0, then the corresponding term, in the left-hand member of (43), is equal to zero. Thus, we examine only the solutions of

$$\nu_1 + \nu_2 + \cdots + \nu_N = Nk_0 + 1, \quad \nu_\ell \ge k_0, \quad 1 \le \ell \le N. \tag{44}$$

Let $\nu_{\ell_1} = \cdots = \nu_{\ell_s} = k_0$ $(s < N)$ and $\nu_j \ge k_0 + 1$, $j \ne \ell_1, \ldots, \ell_s$. In view of (44), we have

$$Nk_0 + 1 \ge sk_0 + (N - s)(k_0 + 1),$$

whence $s \ge N - 1$ and, in fact, $s = N - 1$. Since the right-hand member of (43) is zero, this relation is reduced to $A_{k_0}^{N-1} \cdot A_{k_0+1} = 0$, i.e., $A_{k_0+1} = 0$.

Now we suppose that $A_{k_0+1} = \cdots = A_{k_0+\ell-1} = 0$, $\ell > 1$, and examine the relation (43) with $m = Nk_0 + \ell$. Let us consider the equation

$$\nu_1 + \nu_2 + \cdots + \nu_N = Nk_0 + \ell, \quad \nu_\ell \ge k_0, \quad 1 \le \ell \le N. \tag{45}$$

If $\nu_{\ell_1} = \cdots = \nu_{\ell_r} = k_0$ $(r < N)$, then $\nu_j \ge k_0 + \ell$ for $j \ne \ell_1, \ldots, \ell_r$ (in order to have $A_{\nu_1} \ldots A_{\nu_N} \ne 0$), so that $Nk_0 + \ell \ge rk_0 + (N - r)(k_0 + \ell)$, whence $r = N - 1$ and (43) is reduced to

$$NA_{k_0}^{N-1}\cdot A_{k_0+\ell} = \begin{cases} A_k & \text{if } Nk_0 + \ell = kN \\ 0 & \text{otherwise,} \end{cases}$$

for some integer k. The possibility $Nk_0 + \ell = kN$ implies $k = k_0 + \ell/N$; but

$$k_0 < k_0 + \frac{1}{N} < k_0 + \ell,$$

so that $A_k = 0$ by hypothesis. In both cases, we conclude that $A_{k_0+\ell} = 0$. □

Remarks: The examples

$$f(z) = \frac{z}{(1-\omega)z + \omega}, \quad \omega^q = 1,$$

show that other solutions of (41) are possible for $N = 1$. We may compare (42) with Wedderburn's functional equation $g(x^2) = [g(x)]^2 + 2ax$ (see [1] for references).

5.2 Solution of a Recurrence Relation

There is a relation similar to 1 which may be solved without difficulty. Let A_m, B_m, $m = 1, 2, 3, \ldots$ be two sequences of complex numbers related by

$$(46)\qquad A_m = \sum_{r=1}^{m} \sum_{\substack{\nu_1+\cdots+\nu_r=m \\ \nu_\ell \ge 1}} \prod_{\ell=1}^{r} B_{\nu_\ell}, \quad m = 1, 2, 3, \ldots .$$

We have

$$(47)\qquad B_m = \sum_{r=1}^{m} \sum_{\substack{\nu_1+\cdots+\nu_r=m \\ \nu_\ell \ge 1}} (-1)^{r-1} \prod_{\ell=1}^{r} A_{\nu_\ell}, \quad m = 1, 2, 3, \ldots .$$

Proof: Let

$$f(z) := (1-z)^{-1}, \quad g(z) := \sum_{m=1}^{\infty} B_m z^m.$$

Using Faa di Bruno's formula in the form (20), we obtain

$$\frac{(f(g(z)))^{(m)}(z=0)}{m!} = \sum_{r=1}^{m} \sum_{\substack{\nu_1+\cdots+\nu_r=m \\ \nu_\ell \ge 1}} \prod_{\ell=1}^{r} B_{\nu_\ell} = A_m,$$

whence

$$f(g(z)) = 1 + \sum_{m=1}^{\infty} A_m z^m = \frac{1}{1-g(z)} = \frac{1}{1-\sum_{m=1}^{\infty} B_m z^m}.$$

It follows that

$$\left(1 + \sum_{m=1}^{\infty} A_m z^n\right)\left(1 - \sum_{m=1}^{\infty} B_m z^m\right) \equiv 1,$$

and by comparison of the coefficients:

$$(48)\qquad B_m = A_m - \sum_{s=1}^{m-1} A_{m-s}B_s, \quad m \ge 2.$$

Thus,

$$\begin{aligned} B_m &= A_m - A_{m-1}A_1 - \sum_{s=2}^{m-1} A_{m-s}\left(A_s - \sum_{t=1}^{s-1} A_{s-t}B_t\right) \\ &= A_m - \sum_{s=1}^{m-1} A_{m-s}A_s + \sum_{s=2}^{m-1}\sum_{t=1}^{s-1} A_{m-s}A_{s-t}B_t \\ &= \sum_{\nu_1=m} A_{\nu_1} - \sum_{\nu_1+\nu_2=m} A_{\nu_1}A_{\nu_2} + \sum_{s=2}^{m-1}\sum_{t=1}^{s-1} A_{m-s}A_{s-t}B_t. \end{aligned}$$

At the n^{th} step, we obtain

$$B_m = \sum_{r=1}^{n} (-1)^{r-1} \sum_{\substack{\nu_1 + \cdots + \nu_r = m \\ \nu_\ell \ge 1}} \prod_{\ell=1}^{r} A_{\nu_\ell} + (-1)^n \sum_{s_1=n}^{m-1} \sum_{s_2=n-1}^{s_1-1} \cdots \sum_{s_n=1}^{s_{n-1}-1} A_{m-s_1} \cdots A_{s_{n-1}-s_n} \cdot B_{s_n}, \text{ for } n = 1, 2, \ldots, (m-1).$$

This gives us

$$B_m = \sum_{r=1}^{m-1} (-1)^{r-1} \sum_{\substack{\nu_1 + \cdots + \nu_r = m \\ \nu_\ell \ge 1}} \prod_{\ell=1}^{r} A_{\nu_\ell} + (-1)^{m-1} A_1^{m-1} B_1$$

$$= \sum_{r=1}^{m} (-1)^{r-1} \sum_{\substack{\nu_1 + \cdots + \nu_r = m \\ \nu_\ell \ge 1}} \prod_{\ell=1}^{r} A_{\nu_\ell}. \quad \square$$

5.3 An Identity

Using (32), we can write

$$\frac{\partial}{\partial a}(a \square z) = \lim_{h \to 0} \frac{((a + h) \square z) - (a \square z)}{h} = \lim_{h \to 0} \frac{(h \square (a \square z)) - (a \square z)}{h},$$

and (40) [with $\zeta = h$ and z replaced by $(a \square z)$] gives

(49) $$\frac{\partial}{\partial a}(a \square z) = ((a \square z) - 1)^2 - \frac{1}{2}((a \square z) - 1)^3 + \cdots .$$

On the other hand, (40) gives directly

(50) $$\frac{\partial}{\partial a}(a \square z) = (z - 1)^2 + \left(2a - \frac{1}{2}\right)(z - 1)^3 + \cdots,$$

whence

(51) $$((a \square z) - 1)^2 - \frac{1}{2}((a \square z) - 1)^3 + \cdots$$
$$= (z - 1)^2 + \left(2a - \frac{1}{2}\right)(z - 1)^3 + \cdots, \; z \circlearrowleft 1, \; a \circlearrowleft 0.$$

5.4 An Appearance of the Fibonacci Numbers

The recurrence relation 1 (section 4.1) may be written in the form

(52) $$b_k = \frac{1}{2}a_k - \frac{1}{2}\sum_{r=2}^{k-1} \sum_{\substack{\nu_1 + \cdots + \nu_r = k \\ \nu_\ell \ge 1}} b_r \cdot \prod_{\ell=1}^{r} b_{\nu_\ell}, \quad k \ge 3.$$

To find a bound for $|b_k|(|a_k| \le 1)$, we may try to use (52) with $k = r$, $k = \nu_\ell$ and make the substitutions. To do that, we need to take into account that (52) holds only for $k \ge 3$. In particular, we must examine, separately, the solutions of $\nu_1 + \cdots + \nu_r = k$ with $1 \le \nu_e \le 2$, $1 \le \ell \le r$. This leads us to evaluate the summation

(53) $$\sum_{\frac{k}{2} \le r \le k} \sum_{\substack{\nu_1 + \cdots + \nu_r = k \\ 1 \le \nu_\ell \le 2}} 1 =: \sum_{\frac{k}{2} \le r \le k} p_r(k, 2),$$

where $p_r(k, 2)$ is the number of solutions of $\nu_1 + \cdots + \nu_r = k$, $1 \le \nu_\ell \le 2$. This number is $\binom{r}{k-r}$; indeed, if $\nu_{\ell_1} = \cdots = \nu_{\ell_s} = 1$ and $\nu_\ell = 2$, $\ell \ne \ell_1, \ldots, \ell_s$, then $s \cdot 1 + (r - s) \cdot 2 = k$, so that $s = 2r - k$ and the number of solutions is

$$\binom{r}{s} = \binom{r}{2r - k} = \binom{r}{k - r}$$

(see also the Remark below). Hence, we obtain (see [4], p. 14, Problem 1):

(54) $$\sum_{\frac{k}{2} \le r \le k} p_r(k, 2) = \sum_{\frac{k}{2} \le r \le k} \binom{r}{k - r} = f_k, \quad k = 0, 1, 2, \ldots,$$

the k^{th} Fibonacci number.

Remark: Using the generating function

$$\frac{z^r(z^M - 1)^r}{(z - 1)^r} = \left(\sum_{k=1}^{M} z^k\right)^r = \sum_{k=r}^{rM} p_r(k, M) z^k$$

and the Leibniz formula, we deduce that the number of solutions, $p_r(k, M)$, of the equation $\nu_1 + \cdots + \nu_r = k$, $1 \le \nu_\ell \le M$, is equal to

(55) $$p_r(k, M) = \sum_{j=0}^{\left[\frac{k-r}{M}\right]} (-1)^j \binom{r}{j} \binom{k - jM - 1}{r - 1}, \quad r \le k \le rM.$$

In particular,

$$p_r(k, 2) = \sum_{j=0}^{\left[\frac{k-r}{2}\right]} (-1)^j \binom{r}{j} \binom{k - 2j - 1}{r - 1} = \binom{r}{k - r}, \quad r \le k \le 2r.$$

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References

1. P. Borwein. "Hypertranscendence of the Functional Equation $g(x^2) = [g(x)]^2 + cx$." *Proc. Amer. Math. Soc.* *107.1* (1989):215-21.
2. C. Frappier. "On the Derivatives of Composite Functions." *Fibonacci Quarterly* *25.3* (1987):229-39.
3. R. A. Knoebel. "Exponentials Reiterated." *Amer. Math. Monthly* *88.4* (1981): 235-52.
4. J. Riordan. *An Introduction to Combinatorial Analysis*. New York: Wiley and Sons, 1958.
5. J. Riordan. *Combinatorial Identities*. New York: Wiley and Sons, 1968.

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