ON GENERATING FUNCTIONS FOR POWERS OF RECURRENCE SEQUENCES

Pentti Haukkanen
University of Tampere, P.O. Box 607, SF-33101 Tampere, Finland

Jerzy Rutkowski
Adam Mickiewicz University, ul. Matejki 48/49, Poznań, Poland
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1. Introduction

Let $\{w_q\}$ be a recurrence sequence of order n ($n \in \mathbf{N}$) and let its generating function be given by

$$(1) \qquad w(z) \equiv \sum_{q=0}^{\infty} w_q z^q = \frac{W_1(z)}{\prod_{j=1}^{n}(1 - b_j z)},$$

where $W_1(z)$ is a polynomial in z with deg $W_1(z) = m$. For a positive integer k, let $w_k(z)$ denote the generating function of the sequence $\{w_q^k\}$ of the k^{th} powers of w_q. It is known that $w_k(z)$ is a rational function in z (see [6] or [8]). The aim of this paper is to study the degrees of polynomials in the numerator and denumerator of $w_k(z)$. This paper is similar in character to [4].

The function $w_k(z)$ has been studied with $m = n - 1$ in [8] and [11]. Generating functions for powers of third-order recurrence sequences have been studied in [13], and those of second-order recurrence sequences in [1], [3], [5], [7], [9], [10], and [12].

The proof of our result is based on the following theorem by Hadamard:

If $A(z) = \sum_{n=0}^{\infty} a_n z^n$, $B(z) = \sum_{n=0}^{\infty} b_n z^n$, and $C(z) = \sum_{n=0}^{\infty} a_n b_n z^n$, then

$$C(z) = \frac{1}{2\pi i}\int_{\gamma} A(s)B(z/s)\frac{ds}{s},$$

where γ is a contour in the s plane, which includes the singularities of $B(z/s)/s$ and excludes the singularities of $A(s)$. If the radius of convergence of $A(z)$ [resp. $B(z)$] is R (resp. R'), then the radius of convergence of $C(z)$ is at least RR', and γ may, for example, be any circle between $|s| = R$ and $|s| = |z|/R'$ (see [6], p. 813, [14], pp. 157-59).

2. The Generating Function $w_k(z)$

Theorem: Let $\{w_q\}$ be a recurrence sequence of order n and let its generating function be given by (1). Then

$$(2) \qquad w_k(z) = \frac{W_k(z)}{D_k(z)},$$

where

$$D_k(z) = \prod_{\substack{(r_1, \ldots, r_n) \in \mathbf{N}_0^n \\ r_1 + \cdots + r_n = k}} (1 - b_1^{r_1} \cdots b_n^{r_n} z), \quad \mathbf{N}_0 = \mathbf{N} \cup \{0\},$$

and $W_k(z)$ is a polynomial in z with

$$\deg W_k(z) \le \binom{n + k - 1}{k} - n + m.$$

Proof: Clearly $W_1(z)$ can be written in the form

$$W_1(z) = w_p z^p \prod_{i=1}^{m-p}(1 - a_i z), \quad 0 \le p \le m,$$

where p is the least integer such that $w_p \neq 0$. Assume first that $b_{j_1} \neq b_{j_2}$ for $j_1 \neq j_2$ and $b_j \neq 0$ for $j = 1, 2, \ldots, n$. Then we distinguish two cases: $m < n$, $m \ge n$.

Case 1. Let $m < n$. We proceed by induction on k. If $k = 1$, the theorem holds. Assume it holds for $k = K$ ($K \ge 1$). We shall prove that it holds for $k = K + 1$. Applying Hadamard's theorem and the Cauchy residue theorem and noting that the appropriate winding numbers are = 1, we obtain

$$\begin{aligned} w_{K+1}(z) &= \frac{1}{2\pi i}\int_\gamma w_K(s)w(z/s)\frac{ds}{s} \\ &= \frac{1}{2\pi i}\int_\gamma \frac{W_K(s)w_p z^p \prod_{i=1}^{m-p}(s - a_i z)}{\prod_{r_1+\cdots+r_n=K}(1 - b_1^{r_1}\cdots b_n^{r_n}s)\prod_{j=1}^{n}(s - b_j z)} s^{n-m-1}ds \\ &= \sum_{h=1}^{n} \frac{W_K(b_h z)w_p \prod_{i=1}^{m-p}(b_h - a_i)}{\prod_{r_1+\cdots+r_n=K}(1 - b_1^{r_1}\cdots b_n^{r_n}b_h z)\prod_{\substack{j=1\\ j\neq h}}^{n}(b_h - b_j)} b_h^{n-m-1}. \end{aligned}$$

Denote briefly

$$C_h = w_p \prod_{i=1}^{m-p}(b_h - a_i)\prod_{\substack{j=1\\ j\neq h}}^{n}(b_h - b_j)^{-1} b_h^{n-m-1},$$

$$E_{K+1}^{(h)}(z) = \prod_{r_1+\cdots+r_{h-1}+r_{h+1}+\cdots+r_n=K+1}\left(1 - b_1^{r_1}\cdots b_{h-1}^{r_{h-1}}b_{h+1}^{r_{h+1}}\cdots b_n^{r_n}z\right).$$

Converting the fraction in the sum over h by $E_{K+1}^{(h)}(z)$, we obtain

$$w_{K+1}(z) = \frac{\sum_{h=1}^{n} C_h W_K(b_h z)E_{K+1}^{(h)}(z)}{D_{K+1}(z)}. \tag{3}$$

The number of solutions of the equation $r_1 + \cdots + r_n = K$ in $(r_1, \ldots, r_n) \in \mathbf{N}_0^n$ is equal to

$$\binom{n + K - 1}{K}.$$

Thus, the number of solutions of the equation $r_1 + \cdots + r_{h-1} + r_{h+1} + \cdots + r_n = K + 1$ in $(r_1, \ldots, r_{h-1}, r_{h+1}, \ldots, r_n) \in \mathbf{N}_0^{n-1}$ is equal to

$$\binom{n + K - 1}{K + 1}.$$

This is plainly the degree of the polynomial $E_{K+1}^{(h)}(z)$. Thus, the degree of the polynomial in the numerator of the fraction of (3) is less than or equal to

$$\binom{n + K - 1}{K} - n + m + \binom{n + K - 1}{K + 1},$$

that is, less than or equal to

$$\binom{n + (K + 1) - 1}{K + 1} - n + m.$$

This proves the theorem in Case 1.

Case 2. Let $m \geq n$. We proceed by induction on k in this case, too. The theorem holds for $k = 1$. Assume it holds for $k = K$. Then the series $w_K(z)$ can be written in the form

$$w_K(z) = \sum_{i=0}^{a-b} u_i z^i + \frac{U_K(z)}{D_K(z)},$$

where

$$a = \deg W_K(z) \leq \binom{n + K - 1}{K} - n + m, \quad b = \binom{n + K - 1}{K}$$

and $U_K(z)$ is a polynomial in z of degree $< b$. Note that $a - b \leq m - n$. The series $w(z)$ can be written in the form

$$w(z) = \sum_{j=0}^{m-n} v_j z^j + \sum_{\ell=0}^{n} \frac{A_\ell}{1 - b_\ell z}.$$

Applying Hadamard's theorem and the Cauchy residue theorem and noting that the appropriate winding numbers are =1, we obtain

$$\begin{aligned} w_{K+1}(z) &= \frac{1}{2\pi i}\int_\gamma w_K(s) w(z/s)\frac{ds}{s} \\ &= \frac{1}{2\pi i}\int_\gamma \sum_{i=0}^{a-b}\sum_{j=0}^{m-n} u_i v_j s^i \frac{z^j}{s^{j+1}}\,ds + \frac{1}{2\pi i}\int_\gamma \sum_{i=0}^{a-b}\sum_{\ell=0}^{n} u_i A_\ell \frac{s^i}{s - b_\ell z}\,ds \\ &\quad + \frac{1}{2\pi i}\int_\gamma \sum_{j=0}^{m-n} \frac{U_K(s)}{D_K(s)} v_j \frac{z^j}{s^{j+1}}\,ds + \frac{1}{2\pi i}\int_\gamma \sum_{\ell=0}^{n} \frac{U_K(s)}{D_K(s)} \frac{A_\ell}{s - b_\ell z}\,ds \\ &= \sum_{i=0}^{a-b} u_i v_i z^i + \sum_{i=0}^{a-b}\sum_{\ell=0}^{n} u_i A_\ell b_\ell^j z^i + \sum_{j=0}^{m-n} B_j v_j z^j + \sum_{\ell=0}^{n} \frac{U_K(b_\ell z)}{D_K(b_\ell z)} A_\ell, \end{aligned}$$

where B_j $(j = 0, 1, \ldots, m - n)$ is a complex constant. Now we can see, after some calculations, that $w_{K+1}(z)$ can be written in the form

$$w_{K+1}(z) = \frac{W_{K+1}(z)}{D_{K+1}(z)},$$

where

$$\deg W_{K+1}(z) \leq \binom{n + (K + 1) - 1}{K + 1} - n + m.$$

This proves the theorem in Case 2.

Now the theorem is proved when $b_{j_1} \neq b_{j_2}$ for $j_1 \neq j_2$ and $b_j \neq 0$ for $j = 1, 2, \ldots, n$. But the coefficients of z^q $(q = 0, 1, \ldots)$ in the series $w_k(z)$ and in the polynomials $W_k(z)$ and $D_k(z)$ are polynomials in the variables w_p, a_i, and b_j. Thus, taking limits $b_{j_1} \to b_{j_2}$, $b_j \to 0$ proves that the theorem holds for all $b_1, \ldots, b_n$. This completes the proof.

Remark: It should be noted that, in the case in which two or more of the b_j are equal, the treatment used at the end of the proof does not have to give the best possible result (cf. [8], Sec. 7). However, application of Hadamard's theorem and Cauchy's residue theorem would be too laborious in that case.

Example: Let $\{w_q\} \equiv \{F_q\}$, the Fibonacci sequence, and let $\alpha = (1 + \sqrt{5})/2$, and $\beta = (1 - \sqrt{5})/2$. Then, for $K = 1$, formula (3) is

$$F_2(z) = \frac{\alpha(\alpha - \beta)^{-1}(1 - \beta^2 z) + \beta(\beta - \alpha)^{-1}(1 - \alpha^2 z)}{(1 - \alpha^2 z)(1 - \alpha\beta z)(1 - \beta^2 z)},$$

which gives the well-known formula

$$F_2(z) = \frac{1 - z}{1 - 2z - 2z^2 + z^3}$$

(see, e.g., [2]; [13], p. 794).

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