ϕ-PARTITIONS

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The purpose of this paper is to study partitions of positive integers for which Euler's totient function is endomorphic. That is, $n = a_1 + \cdots + a_i$ is a ϕ-partition if $i \geq 2$, and $\phi(n) = \phi(a_1) + \cdots + \phi(a_i)$.

Questions related to two-summand ϕ-partitions have been considered by the present author [2] and by Makowski [3]; here, we generalize to ϕ-partitions with an arbitrary number of summands. Results include: characterizations of positive integers which have at least one ϕ-partition and of those which have only one ϕ-partition; constructive proof that any prime p has exactly $\pi(p)$ ϕ-partitions; and techniques for constructing ϕ-partitions and reduced ϕ-partitions for various types of positive integers.

Throughout the paper, p and q will denote distinct primes and n will denote a positive integer.

Definition 1: A square-free n is *simple* if $n = 1$ or n has maximal prime divisor p and $q|n$ for every prime $q < p$.

Lemma 2: If s is simple, $n < 2s$, and $n \neq s$, then $\dfrac{s}{\phi(s)} > \dfrac{n}{\phi(n)}$.

Proof: Let $s = 2 \cdot 3 \cdot \cdots \cdot p_i$, and let $2s > n = q_1^{\alpha_1} \cdots q_k^{\alpha_k}$ for $q_1 < \cdots < q_k$. Since $n < 2s$, we have $k \leq i$, and since s is simple, we have $q_j \geq p_j$ for each $1 \leq j \leq k$. If $k = i$ and $q_j = p_j$ for every $1 \leq j \leq k$, then $n = s$. Thus, $k < i$ or $q_j > p_j$ for some $1 \leq j \leq k$. In either case,

$$\frac{n}{\phi(n)} = \frac{q_1 \cdots q_k}{(q_1 - 1) \cdots (q_k - 1)} < \frac{1 \cdot 2 \cdot \cdots \cdot p_i}{1 \cdot 2 \cdot \cdots \cdot (p_i - 1)} = \frac{s}{\phi(s)}.$$

Theorem 3: n has at least one ϕ-partition iff n is not simple.

Proof: (i) Let n be nonsimple. Then there exists a prime p such that $p^\alpha | n$ for $\alpha > 1$, or n is square-free with maximal prime divisor p and there exists $q < p$ such that $q \nmid n$.

Suppose $p^\alpha \| n$ for $\alpha > 1$, and let $n = p^\alpha t$. Then $\phi(n) = \phi(p^\alpha t) = p\phi(p^{\alpha-1}t)$. Hence, $n = \underbrace{p^{\alpha-1}t + \cdots + p^{\alpha-1}t}_{p \text{ summands}}$ is a ϕ-partition.

Now suppose n is square-free with maximal prime divisor p and there exists $q < p$ such that $q \nmid n$. Let $n = pj$ and $p - q = a$. Then

$$\begin{aligned}\phi(pj) &= \phi(p)\phi(j) = (p-1)\phi(j) = (a + q - 1)\phi(j)\\ &= a\phi(j) + (q-1)\phi(j) = a\phi(j) + \phi(qj).\end{aligned}$$

Hence, $n = \underbrace{j + \cdots + j}_{a \text{ summands}} + qj$ is a ϕ-partition.

(ii) Suppose $n = 2 \cdot 3 \cdot \cdots \cdot p_k$ is simple and $n = a_1 + \cdots + a_i$ is a ϕ-partition. Let a_j be a summand of the partition. Since $a_j < n$, it follows from Lemma 2 that

$$\frac{a_j}{\phi(a_j)} < \frac{n}{\phi(n)}.$$

Hence,

$$n = \frac{n}{\phi(n)}\phi(n) = \frac{n}{\phi(n)}\phi(a_1) + \cdots + \frac{n}{\phi(n)}\phi(a_i)$$
$$> \frac{a_1}{\phi(a_1)}\phi(a_1) + \cdots + \frac{a_i}{\phi(a_i)}\phi(a_i) = a_1 + \cdots + a_i.$$

This contradiction completes the proof.

Lemma 4: If $n = a_1 + \cdots + a_i$ is a unique ϕ-partition of n, then each summand is simple.

Proof: Suppose $n = a_1 + \cdots + a_i$ is a unique ϕ-partition and some summand a_j is not simple. Then, by Theorem 3, a_j has a ϕ-partition $a_j = b_1 + \cdots + b_k$; thus, $n = a_1 + \cdots + a_{j-1} + b_1 + \cdots + b_k + a_{j+1} + \cdots + a_i$ is a ϕ-partition of n which is different from $n = a_1 + \cdots + a_i$.

Lemma 5: If a unique ϕ-partition of n has two equal summands, then $n = 2s$ for s simple.

Proof: Suppose $n = s + s + a_1 + \cdots + a_i$ is a unique ϕ-partition of n. If some summand $a_j \neq 0$, then $n = 2s + a_1 + \cdots + a_i$ is a different ϕ-partition of n. Therefore, each $a_j = 0$ and $n = 2s$. By Lemma 4, s is simple.

Theorem 6: n has a unique ϕ-partition iff $n = 2s$ for s simple or $n = 3$.

Proof: (i) Suppose n has a unique ϕ-partition. Then, by Theorem 3, n is not simple.

If n is square-free with maximum prime divisor p and $q < p$ such that $q \nmid n$, let $n = pj$ and $p - q = a$. Then, from the proof of Theorem 3(i), we have

$$n = \underbrace{j + \cdots + j}_{a \text{ summands}} + qj \text{ is a } \phi\text{-partition.}$$

And since it is unique, Lemma 4 implies that j is simple and Lemma 5 implies that $a = 1$. Thus, $p - q = 1$. Hence, we have $p = 3$, $q = 2$, and $n = 3$.

Now suppose $p^{\alpha} \| n$ for $\alpha > 1$ and $n = p^{\alpha}t$. Then

$$n = \underbrace{p^{\alpha-1}t + \cdots + p^{\alpha-1}t}_{p \text{ summands}} \text{ is a } \phi\text{-partition,}$$

and since it is unique, we have that $p^{\alpha-1}t$ is simple (Lemma 4). Therefore, by Lemma 5, $n = 2s$ for s simple.

(ii) It is obvious that $3 = 1 + 2$ is a unique ϕ-partition of 3.

Let $n = 2s$ for s simple. Clearly, $2s = s + s$ is a ϕ-partition. Suppose $2s = a_1 + \cdots + a_i$ is a different ϕ-partition. Then there exists a summand $a_j \neq s$. Since $a_j < 2s$, we have, by Lemma 2, that

$$\frac{a_j}{\phi(a_j)} < \frac{s}{\phi(s)}.$$

This gives the contradiction,

$$2s = \frac{2s\phi(s)}{\phi(s)} = \frac{s\phi(2s)}{\phi(s)} = \frac{s}{\phi(s)}(\phi(a_1) + \cdots + \ (a_i))$$
$$= \frac{s}{\phi(s)}\phi(a_1) + \cdots + \frac{s}{\phi(s)}\phi(a_i) > \frac{a_1}{\phi(a_1)}\phi(a_1) + \cdots + \frac{a_i}{\phi(a_i)}\phi(a_i)$$
$$= a_1 + \cdots + a_i.$$

Hence, $2s = s + s$ is a unique ϕ-partition of n.

Theorem 7: $p = a_1 + \cdots + a_i$ is a ϕ-partition iff one summand is prime and every other summand is 1.

Proof: (i) $p = \underbrace{1 + \cdots + 1}_{p-q \text{ summands}} + q$ is clearly a ϕ-partition for every prime $q < p$.

(ii) Let $p = a_1 + \cdots + a_i$ be a ϕ-partition. It is obvious that at least one summand is greater than 1. Suppose the two summands, a_1 and a_2, are each greater than 1. Then $\phi(a_1) \leq a_1 - 1$ and $\phi(a_2) \leq a_2 - 1$. Therefore, we have the contradiction

$$\begin{aligned} a_1 + \cdots + a_i - 1 = p - 1 &= \phi(p) \\ &= \phi(a_1) + \cdots + \phi(a_i) \leq a_1 + \cdots + a_i - 2. \end{aligned}$$

Assume $a_1 > 1$. Then $a_1 = p - i + 1$, and

$$p - 1 = \phi(p) = \underbrace{\phi(1) + \cdots + \phi(1)}_{i-1 \text{ summands}} + \phi(a_1) = i - 1 + \phi(a_1).$$

Hence, $\phi(a_1) = p - i = a_1 - 1$. Therefore, a_1 is prime.

As an immediate consequence of this theorem, we get

Corollary 8: A prime p has exactly $\pi(p)$ ϕ-partitions.

We now provide two very general techniques for constructing ϕ-partitions for a particular n.

1. If n is even, $p \| n$, $p = 2^{a_1} + \cdots + 2^{a_i} + q$, $q \nmid n$, and $n = 2^{\alpha}pm$, then $n = 2^{a_1+\alpha}m + \cdots + 2^{a_i+\alpha}m + 2^{\alpha}mq$ is a ϕ-partition.

Some results regarding how many ways a particular prime p can be written as the sum of a prime and powers of 2 are given in [1].

Definition 9: A positive integer m is *prime dependent* on n if every prime divisor of m is a divisor of n.

Notice that if m is prime dependent on n then $\phi(mn) = m\phi(n)$.

2. If $n = p^{\alpha}t$ where $\alpha > 1$ and $p \nmid t$, and $p = a_1 + \cdots + a_i$ such that each summand is prime dependent on n, then

$$n = a_1 p^{\alpha-1}t + \cdots + a_i p^{\alpha-1}t \text{ is a } \phi\text{-partition.}$$

Notice that for every p such that $p^{\alpha} | n$ for $\alpha > 1$ we get a ϕ-partition of n with p summands by letting

$$p = \underbrace{1 + \cdots + 1}_{p \text{ summands}}$$

in construction 2. If n is even, for each such p we can get ϕ-partitions with x summands for every x satisfying $a \leq x \leq p$, where a is the number of nonzero digits in the binary representation of p.

Definition 10: If $n = a_1 + \cdots + a_i$ and $a_1 = b_1 + \cdots + b_j$ are ϕ-partitions, then $n = b_1 + \cdots + b_j + a_2 + \cdots + a_i$ is an expansion of $n = a_1 + \cdots + a_i$.

Expansions are clearly ϕ-partitions.

Definition 11: A ϕ-partition is *reduced* if each of its summands is simple.

It is obvious that a ϕ-partition can be expanded iff it is not reduced. So every nonsimple number has at least one reduced ϕ-partition. The following are examples of reduced ϕ-partitions for various types of n:

(i) $2^{a} = \underbrace{2 + \cdots + 2}_{2^{a-1} \text{ summands}}$

(ii) $p^{\alpha} = \underbrace{1 + \cdots + 1}_{p^{\alpha-1}(p-2) \text{ summands}} + \underbrace{2 + \cdots + 2}_{p^{\alpha-1} \text{ summands}}$

(iii) $2^a p^\alpha = \underbrace{2 + \cdots + 2}_{2^{a-1}p^{\alpha-1}(p-3)\text{ summands}} + \underbrace{6 + \cdots + 6}_{2^{a-1}p^{\alpha-1}\text{ summands}}$

(iv) $pq = \underbrace{1 + \cdots + 1}_{(p-2)(q-2)\text{ summands}} + \underbrace{2 + \cdots + 2}_{p+q-5\text{ summands}} + 6$

Several open questions about two-summand ϕ-partitions could be resolved if it can be shown that reduction is unique. Evidence and intuition strongly suggest that it is; but it seems that a proof may be quite difficult. We close with the conjecture: Every nonsimple number has exactly one reduced ϕ-partition.

References

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2. Patricia Jones. "On the Equation $\phi(x) + \phi(k) = \phi(x + k)$." *Fibonacci Quarterly* *28.2* (1990):162-65.
3. A. Makowski. "On Some Equations Involving Functions $\phi(n)$ and $\sigma(n)$." *Amer. Math. Monthly* *67* (1960):668-70.

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