SECOND-ORDER STOLARSKY ARRAYS

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In 1977, Kenneth B. Stolarsky [6] introduced an array $s(i, j)$ of positive integers such that every positive integer occurs exactly once in the array, and every row satisfies the familiar Fibonacci recurrence:

$$s(i, j) = s(i, j - 1) + s(i, j - 2) \text{ for all } j \geq 3 \text{ for all } i \geq 1.$$

The first seven rows of Stolarsky's array begin as shown here:

1	2	3	5	8	13	21	...
4	6	10	16	26	42	68	...
7	11	18	29	47	76	123	...
9	15	24	39	63	102	165	...
12	19	31	50	81	131	212	...
14	23	37	60	97	157	254	...
17	28	45	73	118	191	309	...

Hendy [4], Butcher [2], and Gbur [3] considered Stolarsky's array, and Morrison [5] and Burke and Bergum [1, p. 146] considered closely related arrays. In particular, Gbur discussed arrays whose row recurrence is given by

$$s(i, j) = as(i, j - 1) + s(i, j - 2),$$

which, for $a = 1$, is the row recurrence for Stolarsky's original array. In this note, we show that any one of a larger class of second-order recurrences can be used to construct infinitely many Stolarsky arrays.

Define a *Stolarsky pre-array (of q rows)* as an array $s(i, j)$ of distinct positive integers satisfying

$$s(i, j) = as(i, j - 1) + bs(i, j - 2) \text{ for all } j \geq 3 \text{ for } 1 \leq i \leq q,$$

where a and b are integers satisfying $1 \leq b \leq a$, and the numbers 1, 2, 3, ..., q are all present in the array. By a *Stolarsky array* we shall mean an array $s(i, j)$ whose first q rows comprise a Stolarsky pre-array for every positive integer q. For the following Stolarsky pre-array, $q = 2$, $a = 1$, and $b = 1$:

1	4	5	9	12	23	37	60	...
2	8	10	18	28	46	74	120	...

In order to construct Row 3 beginning with $s(3, 1) = 3$, note that $s(3, 2)$ cannot be 4 or 5, as these appear in Row 1; nor 6, as then $s(3, 3) = 9$, already in Row 1; nor 7 nor 8 nor 9 nor 10 nor 11. These observations illustrate the problem: *once q rows of a (prospective) Stolarsky array have been constructed, can Row $q + 1$ always be constructed*? We shall show that the answer is *yes*, and that, actually, Row $q + 1$ can be constructed in infinitely many ways.

The symbols s_1, s_2, ... will always represent a sequence of the following kind:

(i) $s_1 > 0$, $s_2 > 0$, and $s_n = as_{n-1} + bs_{n-2}$ for $n \geq 3$,

where a and b are integers satisfying $1 \leq b \leq a$. Let

$$\alpha = \frac{a + \sqrt{a^2 + 4b}}{2} \quad \text{and} \quad \beta = a - \alpha,$$

so that $\alpha > 1$, $-1 < \beta < 0$, and the identities $\alpha^2 = a\alpha + b$ and $\beta^2 = a\beta + b$ yield

(ii) $s_n = a_1\alpha^n + b_1\beta^n$ for all $n \geq 1$, where

$$a_1 = \frac{s_1\beta - s_2}{\alpha(\beta - \alpha)} \quad \text{and} \quad b_1 = \frac{s_2 - s_1\alpha}{\beta(\beta - \alpha)}.$$

Similarly, the symbols $t_1, t_2, \ldots$ will always mean a sequence given by

$$t_n = at_{n-1} + bt_{n-2} = a_2\alpha^n + b_2\beta^n,$$

where

$$a_2 = \frac{t_1\beta - t_2}{\alpha(\beta - \alpha)} \quad \text{and} \quad b_2 = \frac{t_2 - t_1\alpha}{\beta(\beta - \alpha)}, \quad \text{and } t_1 > 0,\ t_2 > 0.$$

Lemma 1.1: There exists a positive integer N such that $s_{n+1} = [\alpha s_n + \frac{1}{2}]$ for every $n \geq N$. The least such N is $2 + [\log_{\alpha/b} 2|\alpha s_1 - s_2|]$.

Proof: $\alpha s_n = \alpha(a_1\alpha^n + b_1\beta^n) = a_1\alpha^{n+1} + b_1\beta^{n+1} + \alpha b_1\beta^n - b_1\beta^{n+1}$

$$= s_{n+1} + b_1\beta^n(\alpha - \beta),$$

so that $s_{n+1} = [\alpha s_n + \frac{1}{2}]$ if and only if $0 < b_1\beta^n(\alpha - \beta) + \frac{1}{2} < 1$. This is equivalent to $-1 < 2(\alpha s_1 - s_2)\beta^{n-1} < 1$, hence to

$$\left(\frac{b}{\alpha}\right)^{n-1} = |\beta^{n-1}| < \frac{1}{2|\alpha s_1 - s_2|},$$

and hence equivalent to $n - 1 \geq \log_{\alpha/b} 2|\alpha s_1 - s_2|$, as required.

Lemma 1.2: Suppose s_1 is not among $t_1, t_2, \ldots$, and t_1 is not among $s_1, s_2, \ldots$. Let

$$M = 2 + [\log_{\alpha/b} 2|\alpha s_1 - s_2|] \quad \text{and} \quad N = 2 + [\log_{\alpha/b} 2|\alpha t_1 - t_2|].$$

If $m \geq M$, $n \geq N$, and $s_m < t_n \leq s_{m+1}$, then $s_m < t_n < s_{m+1} < t_{n+1} < s_{m+2} < \cdots$.

Proof: Suppose $m \geq M$ and $n \geq N$. By Lemma 1.1, $s_{i+1} = [\alpha s_i + \frac{1}{2}]$ for every $i \geq m$ and $t_{i+1} = [\alpha t_i + \frac{1}{2}]$ for every $i \geq n$. So, if $t_n = s_{m+1}$, then

$$[\alpha t_n + \tfrac{1}{2}] = [\alpha s_{m+1} + \tfrac{1}{2}],$$

so that $t_{n+1} = s_{m+2}$. But then $at_n + bt_{n-1} = as_{m+1} + bs_m$, so that $t_{n-1} = s_m$. But then $at_{n-1} + bt_{n-2} = as_m + bs_{m-1}$, so that $t_{n-2} = s_{m-1}$. Continuing, we eventually reach $t_1 = s_p$ for some $p \geq 1$ or else $t_q = s_1$ for some $q \geq 1$, contrary to the hypothesis.

Now that we have $s_m < t_n$ and $t_n < s_{m+1}$, the remaining inequalities in the asserted chain follow by induction: $s_p < t_q$ implies

$$[\alpha s_p + \tfrac{1}{2}] < [\alpha t_q + \tfrac{1}{2}],$$

so that $s_{p+1} < t_{q+1}$, and $t_q < s_r$ similarly implies $t_{q+1} < s_{r+1}$.

Lemma 1.3: Suppose s_1, s_2, and t_1 are given and $t_1 > s_1$. For $k \geq 1$, let $t_j^{(k)}$ denote the sequence t_1, $t_2 = t_1 + k$, $t_3 = at_2 + bt_1$, $\ldots$. Then there exist positive integers C and K, both independent of k, such that if $k > K$ and $m > C[\log_\alpha k]$ and n is the index satisfying $s_m < t_n^{(k)} \leq s_{m+1}$, then

$$s_m < t_n^{(k)} < s_{m+1} < t_{n+1}^{(k)} < s_{m+1} < \cdots .$$

Proof: Let

$$M = 2 + [\log_{\alpha/b} 2|\alpha s_1 - s_2|] \quad \text{and} \quad N(k) = 2 + [\log_{\alpha/b} 2|\alpha t_1 - t_1 - k|].$$

Let $p(k)$ be the index satisfying

$$s_{p(k)} < t_{N(k)}^{(k)} \leq s_{p(k)+1}.$$

Clearly, there is a positive integer K_1 so large that $p(k) \geq M$ for all $k \geq K_1$. For such k, Lemma 1.2 gives

(1) $s_{p(k)+h} < t_{N(k)+h}^{(k)} < s_{p(k)+1+h}$ for all $h \geq 0$.

Also, for all $k \geq K_1$,

$$a_1\alpha^{p(k)} + b_1\beta^{p(k)} = s_{p(k)} < t_{N(k)}^{(k)} = a_2\alpha^{N(k)} + b_2\beta^{N(k)} < (a_2 + |b_2|)\alpha^{N(k)}.$$

Let A, B, K_2 be positive integers, with $K_2 > K_1$, all independent of k, satisfying $a_2 + |b_2| < A + Bk$ for all $k > K_2$; to see that such A and B exist, observe

$$a_2 = \frac{t_1\beta - (t_1 + k)}{\alpha(\beta - \alpha)} \quad \text{and} \quad b_2 = \frac{t_1 + k - t_1\alpha}{\beta(\beta - \alpha)}.$$

For all such k,

$$a_1\alpha^{p(k)} < (A + Bk)\alpha^{N(k)} + Q(k), \text{ where } Q(k) = 1 + |b_1\beta^{p(k)}|.$$

Then

$$a_1\alpha^{p(k)} < Q(k) + (A + Bk)\alpha^{2+\log_{\alpha/b} 2|\alpha t_1 - t_1 - k|},$$

so that

$$a_1\alpha^{p(k)} < Q(k) + \alpha^2(A + Bk)(2|\alpha t_1 - t_1 - k|)^{\frac{1}{1 - \log_\alpha b}}.$$

Applying $\log_\alpha$ to both sides and the inequality $\log_\alpha(x + y) < \log_\alpha x + \log_\alpha y$ to the resulting right-hand side yields

$$p(k) + \log_\alpha a_1 < \log_\alpha Q(k) + 2 + \log_\alpha(A + Bk) + \frac{1}{1 - \log_\alpha b}\log_\alpha(2|\alpha t_1 - t_1 - k|).$$

Now $\lim_{k\to\infty} Q(k) = 1$, so that there must exist positive integers C and K_3, independent of k, with $K_3 > K_2$, such that

$$p(k) + 1 < C[\log_\alpha k] \text{ for all } k > K_3.$$

For such k, if m is any integer that exceeds $C[\log k]$, then $m = p(k) + h$ for some $h \geq 1$. For $n = N(k) + m - p(k)$, the stated chain of inequalities follows from (1).

Theorem: Let $S = \{s(x, y): 1 \leq x \leq q, y \geq 1\}$ be a Stolarsky pre-array. Suppose $t_1 \notin S$ and $t_1 > \max\{s(x, 1): 1 \leq x \leq q\}$. Then there exist infinitely many numbers t_2 such that no term of the sequence $t_1, t_2, t_3 = at_2 + bt_1, \ldots$ lies in S.

Proof: Suppose, to the contrary, that there are at most finitely many numbers $k \geq 1$ for which the sequence t_1, $t_2 = t_1 + k$, $t_3 = at_2 + bt_1$, ... contains no element of S. Let k_1 be the greatest of these k. Let $t_1^{(k)}, t_2^{(k)}, \ldots$ denote the (a, b)-recurrence sequence whose first two terms are t_1 and $t_2 = t_1 + k_1 + k$. Then, for every positive integer k, the sequence $t_1^{(k)}, t_2^{(k)}, \ldots$ contains a term of S. That is, there exist indices $j(k)$, $x(k)$, and $y(k)$ for which

(2) $t_{j(k)}^{(k)} = s(x(k), y(k))$, where

(3) $1 \leq x(k) \leq q$.

On the other hand, by Lemma 1.3, there exist constants $C_1, C_2, \ldots, C_q$ and $K_1, K_2, \ldots, K_q$, all independent of k, such that for $x = 1, 2, \ldots, q$, if

$$y_x > C_x[\log_\alpha k]$$

where $k > K_x$ and j_x is the index for which

$$s(x, y_x) < t_{j_x}^{(k)} \leq s(x, y_x + 1),$$

then equation (2) cannot hold for any $j(k) \leq j_x$. Accordingly, (2) implies

(4) $\quad 1 \le y(k) \le C_{x(k)}[\log k]$ for all $k > K = \max\{K_1, K_2, \ldots, K_q\}$.

Now, since the index $x(k)$ in (2) is $\le q$, we have $s(x(k), 1) < t_1^{(k)}$ for all k, by hypothesis, and also $s(x(k), 2) < t_2^{(k)}$ for all k larger than some K^*. Therefore, in equation (2), $j(k) \le y(k)$, so that

(5) $\quad 1 \le j(k) \le C_{x(k)}[\log_\alpha k]$ for all $k > K^*$.

Let $m(k) = [\log_\alpha k] \max\{C_1, C_2, \ldots, C_q\}$. Then, for all $k > \mathbb{K} = \max\{K, K^*\}$, we have

$$1 \le x(k) \le q,\ 1 \le y(k) \le m(k),\ 1 \le j(k) \le m(k).$$

Let k' be any integer large enough that $k' > q[m(\mathbb{K} + k')]^2$. Then, for $k = 1, 2, 3, \ldots, k'$, we have

$$1 \le x(\mathbb{K} + k) \le q,\ 1 \le y(\mathbb{K} + k) \le m(\mathbb{K} + k'),\ 1 \le j(\mathbb{K} + k) \le m(\mathbb{K} + k').$$

Now, the total number of *distinct* triples (x, y, j) that can satisfy three such inequalities is the product $q[m(\mathbb{K} + k')]^2$, but we have more than this number. Therefore, there exist distinct k_u and k_v for which

$$x(k_u) = x(k_v),\ y(k_u) = y(k_v),\ j(k_u) = j(k_v).$$

This means that the sequences

$$t_1,\ t_2^{(k_u)},\ \ldots,\ t_{j(k_u)}^{(k_u)},\ \ldots \quad \text{and} \quad t_1,\ t_2^{(k_v)},\ \ldots,\ t_{j(k_v)}^{(k_v)},\ \ldots$$

have identical first terms and identical $j(k_u)^{\text{th}}$ terms. But this implies

$$t_2^{(k_u)} = t_2^{(k_v)},$$

contrary to $k_u \ne k_v$. This contradiction finishes the proof.

Conclusion

An obvious consequence of the theorem is that any Stolarsky pre-array can be extended to a Stolarsky array. For each new row, one need only choose t_1 to be the *least* positive integer satisfying the hypothesis of the theorem; that is, the least not yet present in the array being constructed. This choice ensures that every positive integer must occur in the constructed Stolarsky array.

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