ARITHMETIC SEQUENCES AND FIBONACCI QUADRATICS

Mahesh K. Mahanthappa
Southern Hills Junior High School, Boulder, CO 80303
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1. Introduction

It is known [1] that the equation $F_n x^2 + F_{n+1}x - F_{n+2} = 0$ has solutions -1 and F_{n+2}/F_n, where $\{F_n\}_{n \geq 1}$ denotes the Fibonacci sequence. One wonders if other interesting results might be obtained if the coefficients of the quadratic equation were some other functions of the Fibonacci numbers. The answer, as might be expected, is in the affirmative. Surprisingly, however, the results in this paper arise in response to the following quite different question. Under what conditions does the quadratic equation $ax^2 + bx - c = 0$ have rational roots given that a, b, and c are represented by the arithmetic sequence n, $n + r$, $n + 2r$ in some order, where n and r are positive integers? In this paper, we treat only the case $r = 1$.

As usual, $\{L_n\}_{n \geq 1}$ will denote the Lucas sequence and α the golden ratio. Moreover, we will have occasion to use such well-known results as

$$L_n = F_{n+1} + F_{n-1},\ L_n + F_n = 2F_{n+1},\ L_n - F_n = 2F_{n-1},\ \alpha^n = (L_n + F_n\sqrt{5})/2$$

(see [2]). Note that $L_n = F_{n+1} + F_{n-1}$ can be written as

(1) $\quad L_n = 2F_{n-1} + F_n.$

Also, we will need the following identities from [2]:

(2a) $\quad F_{n+1}^2 = F_n F_{n+2} + (-1)^n;$

(2b) $\quad F_{n+1}F_{n-2} = F_n F_{n-1} + (-1)^{n+1}.$

2. Fibonacci Quadratics

The equations

$$ax^2 + bx - c = 0,\quad ax^2 - bx - c = 0,$$
$$cx^2 + bx - a = 0,\quad \text{and } cx^2 - bx - a = 0$$

have the same discriminant. Therefore, we shall study only the first one. Let us consider the case $r = 1$.

Theorem 1: Rational solutions to

(3) $\quad nx^2 + (n + 1)x - (n + 2) = 0$

exist *if and only if*

(4a) $\quad n = F_{2m+1} - 1 \quad (m \geq 1)$

and they are

(4b) $\quad F_{2m}/(F_{2m+1} - 1),\quad -F_{2m+2}/(F_{2m+1} - 1).$

Proof: The discriminant of (3) is

$$D_1 = (n + 1)^2 + 4n(n + 2)$$
$$= 5(n + 1)^2 - 4.$$

Rational solutions of (3) exist if and only if D_1 is a perfect square, say, for example, $D_1 = t^2$. Then we have

(4c) $t^2 - 5(n+1)^2 = -4$,

which has positive solutions $t = L_{2m+1}$ and $n = F_{2m+1} - 1$ with $m \ge 1$ for $n \ne 0$, as shown by Long and Jordan [4, Lemma 1], although their proof can be considerably simplified by the use of the identity $\alpha_n = (L_n + F_n\sqrt{5})/2$. But, by (1), $t = 2F_{2m} + F_{2m+1}$ and $b = n + 1 = F_{2m+1}$. Using these values in

$$x = (-b \pm t)/2n,$$

we get (4b). It is interesting to note that the solutions are proportional to F_{2m} and F_{2m+2}, which precede and follow F_{2m+1}, respectively.

Theorem 2: Rational solutions to

(5) $nx^2 + (n+2)x - (n+1) = 0$

exist *if and only if*

(6a) $n = F_{2m+3}F_{2m} \quad (m \ge 1)$

and they are

(6b) $F_{2m+2}/F_{2m+3}, \quad -F_{2m+1}/F_{2m}$.

Proof: The discriminant of (5) is

$$\begin{aligned} D_2 &= (n+2)^2 + 4n(n+1) \\ &= n^2 + 4(n+1)^2. \end{aligned}$$

Rational solutions of (5) exist if and only if D_2 is a perfect square, $D_2 = t^2$. Thus, $[n, 2(n+1), t]$ form a Pythagorean triplet, not necessarily primitive. We represent the triplet as $(g^2 - h^2, 2gh, g^2 + h^2)$ to get

(6c) $g^2 - gh - (h^2 - 1) = 0$.

[Note that if it were represented as $(2gh, g^2 - h^2, g^2 + h^2)$ then $g^2 - h^2 = 4gh + 2$ and this implies $g^2 - h^2 \equiv 2 \pmod 4$, an impossibility.] But, again, g is an integer if and only if the discriminant of (6c) is a perfect square:

$$h^2 + 4(h^2 - 1) = 5h^2 - 4 = s^2$$

or

(6d) $s^2 - 5h^2 = -4$.

This is the same Pell equation as before and so has solutions $s = L_{2m+1}$ and $h = F_{2m+1}$. Now

$$g = (h \pm s)/2 = [F_{2m+1} \pm L_{2m+1}]/2 = (F_{2m+1} + F_{2m}), -F_{2m} = F_{2m+2}, -F_{2m}.$$

Since only the first solution gives positive n,

$$n = g^2 - h^2 = F^2_{2m+2} - F^2_{2m+1} = F_{2m+3}F_{2m},$$

with $m \ge 1$, for $n \ne 0$. In this case, using (2b) and (2a), we obtain

$$\begin{aligned} b = F_{2m+3}F_{2m} + 2 &= F_{2m+2}F_{2m+1} + 1 = F_{2m+2}(F_{2m+2} - F_{2m}) + 1 \\ &= F^2_{2m+2} - F_{2m+2}F_{2m} + 1 = F_{2m+3}F_{2m+1} - F_{2m+2}F_{2m} \end{aligned}$$

and

$$t = g^2 + h^2 = F^2_{2m+2} + F^2_{2m+1} = F_{2m+3}F_{2m+1} + F_{2m+2}F_{2m}.$$

Using these in $x = (-b \pm t)/2n$, we obtain the solutions (6b) as claimed.

The last equation to be considered is

$$(n + 1)x^2 + nx - (n + 2) = 0.$$

Instead, we investigate the equivalent equation

$$nx^2 + (n - 1)x - (n + 1) = 0.$$

Theorem 3: Rational solutions to

(7) $\quad nx^2 + (n - 1)x - (n + 1) = 0$

exist *if and only if*

(8a) $\quad n = F_{2m+1}F_{2m} \quad (m \geq 1)$

and they are

(8b) $\quad F_{2m-1}/F_{2m}, \quad -F_{2m+2}/F_{2m+1}.$

Proof: The discriminant of (7) is

$$D_3 = (n - 1)^2 + 4n(n + 1) = 4n^2 + (n + 1)^2.$$

Rational solutions of (7) exist if and only if D_3 is a perfect square, $D_3 = t^2$. Thus, $(2n, n + 1, t)$ form a Pythagorean triplet. We represent the triplet as $(2gh, g^2 - h^2, g^2 + h^2)$ to get

(8c) $\quad g^2 - gh - (h^2 + 1) = 0.$

[Note that if it were represented as $(g^2 - h^2, 2gh, g^2 + h^2)$ then we would have $4gh - 2 = g^2 - h^2$ and this implies $g^2 - h^2 \equiv 2 \pmod 4$, an impossibility.] As before, g is an integer if and only if the discriminant of (8c) is a perfect square:

$$h^2 + 4(h^2 + 1) = 5h^2 + 4 = s^2$$

or

(8d) $\quad s^2 - 5h^2 = 4$

which has positive solutions $s = L_{2m}$ and $h = F_{2m}$ for $m \geq 1$ by [4, Lemma 2]. Since

$$g = (h \pm s)/2 = (F_{2m} \pm L_{2m})/2 = (F_{2m} + F_{2m-1}), -F_{2m-1} = F_{2m+1}, -F_{2m-1}.$$

Only the first solution gives positive n:

$$n = gh = F_{2m+1}F_{2m}$$

with $m \geq 1$, for $n \neq 0$. In this case, using (2a) and (2b), we have that

$$b = F_{2m+1}F_{2m} - 1 = F_{2m}(F_{2m+2} - F_{2m}) - 1 = F_{2m+2}F_{2m} - (F_{2m}^2 + 1)$$
$$= F_{2m+2}F_{2m} - F_{2m+1}F_{2m-1}$$

and

$$t = g^2 + h^2 = F_{2m+1}^2 + F_{2m}^2 = F_{2m+2}F_{2m} + F_{2m+1}F_{2m-1}.$$

Using these in $x = (-b \pm t)/2n$, we obtain the solutions (8b) as claimed.

The case $r > 1$ is under consideration.

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