GENERALIZED MULTIVARIATE FIBONACCI POLYNOMIALS OF ORDER K AND THE MULTIVARIATE NEGATIVE BINOMIAL DISTRIBUTIONS OF THE SAME ORDER

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1. Introduction and Summary

In a recent paper, Philippou and Antzoulakos [4] introduced and studied the sequence of multivariate Fibonacci polynomials of order k and related them to the multiparameter negative binomial distribution of the same order of Philippou [3], in order to derive a recurrence relation for calculating its probabilities. This sequence of polynomials includes, as a special case, both the sequence of Fibonacci polynomials of order k and the sequence of Fibonacci-type polynomials of the same order of Philippou, Georghiou, and Philippou [9] and [10], respectively.

In this paper, we introduce a generalization of the sequence of multivariate Fibonacci polynomials of order k (see Definition 2.1), and we derive an expansion in terms of the multinomial coefficients and a recurrence for the general term of the $(r - 1)$-fold convolution of this sequence with itself (see Theorems 2.1 and 2.2). Next, we relate these polynomials to the multivariate negative binomial distribution of order k of Philippou, Antzoulakos, and Tripsiannis [8], and we derive a useful recurrence relation for calculating its probabilities (see Proposition 3.1 and Theorem 3.1). Analogous recurrences follow directly for the type I, type II, and extended multivariate negative binomial distributions of order k of [8] (see Corollaries 3.1-3.3).

The present paper generalizes results on multivariate Fibonacci polynomials of order k (see Remark 2.1) and Fibonacci-type polynomials of the same order (see Remark 2.2). At the same time, several results of Aki [1], Philippou and Georghiou [6], and Philippou and Antzoulakos [4] on recurrences for the probabilities of univariate geometric and negative binomial distributions of order k are generalized to the multivariate case.

Unless otherwise stated, in this paper k, m, and r are fixed positive integers, n_i $(1 \le i \le m)$ are integers, n_{ij} $(1 \le i \le m$ and $1 \le j \le k)$ are nonnegative integers as specified, x_{ij} $(1 \le i \le m$ and $1 \le j \le k)$ are real numbers in the interval $(0, \infty)$, $\underline{1}$ denotes the m-dimensional vector with a one in every position, and $\underline{j}_i$ $(1 \le i \le m$ and $1 \le j \le k)$ denotes the m-dimensional vector with a j in the i^{th} position and zeros elsewhere. Also, whenever sums and products are taken over i and j, ranging, respectively, from 1 to m and from 1 to k, we shall omit these limits for notational simplicity.

2. Generalized Multivariate Fibonacci Polynomials of Order k and Convolutions

In this section, we introduce the sequence of generalized multivariate Fibonacci polynomials of order k, to be denoted by

$$H_{\underline{n}}^{(k)}(\underline{x}_1, \ldots, \underline{x}_m),$$

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along with the $(r - 1)$-fold convolution of $H_{\underline{n}}^{(k)}(\underline{x}_1, \ldots, \underline{x}_m)$ with itself, to be denoted by

$$H_{\underline{n},r}^{(k)}(\underline{x}_1, \ldots, \underline{x}_m),$$

and we derive a multinomial expansion and a recurrence for the $\underline{n}^{\text{th}}$ term of $H_{\underline{n},r}^{(k)}(\underline{x}_1, \ldots, \underline{x}_m)$. In some instances, we shall use the notation $H_{\underline{n}}^{(k)}$ and $H_{\underline{n},r}^{(k)}$ instead of $H_{\underline{n}}^{(k)}(\underline{x}_1, \ldots, \underline{x}_m)$ and $H_{\underline{n},r}^{(k)}(\underline{x}_1, \ldots, \underline{x}_m)$, respectively.

Definition 2.1: The sequence of polynomials $H_{\underline{n}}^{(k)}(\underline{x}, \ldots, \underline{x}_m)$ is said to be the sequence of generalized multivariate Fibonacci polynomials of order k, if

$$H_{\underline{n}}^{(k)}(\underline{x}_1, \ldots, \underline{x}_m) = \begin{cases} 0, & \text{if some } n_i \le 0 \ (1 \le i \le m), \\ 1, & \text{if } \underline{n} = \underline{1}, \\ \sum_i \sum_j x_{ij} H_{\underline{n}-\underline{j}_i}^{(k)}(\underline{x}_1, \ldots, \underline{x}_m), & \text{elsewhere,} \end{cases}$$

where $\underline{n} = (n_1, \ldots, n_m)$ and $\underline{x}_i = (x_{i1}, \ldots, x_{ik})$, $i = 1, \ldots, m$.

For $m = 1$, $n_1 = n$ (≥ 0) and $\underline{x}_1 = \underline{x}$, $H_{\underline{n}}^{(k)}(\underline{x}_1, \ldots, \underline{x}_m)$ reduces to $H_n^{(k)}(\underline{x})$, the sequence of multivariate Fibonacci polynomials of order k of Philippou and Antzoulakos [4].

Lemma 2.1: Let $H_{\underline{n}}^{(k)}(\underline{x}_1, \ldots, \underline{x}_m)$ be the sequence of generalized multivariate Fibonacci polynomials of order k, and denote its generating function by

$$g_k(t_1, \ldots, t_m; \underline{x}_1, \ldots, \underline{x}_m).$$

Then, for $0 < x_{ij} < 1$ $(1 \le i \le m$ and $1 \le j \le k)$ and $\sum_i \sum_j x_{ij} < 1$, we have

$$g_k(t_1, \ldots, t_m; \underline{x}_1, \ldots, \underline{x}_m) = \frac{t_1 \ldots t_m}{1 - \sum_i \sum_j x_{ij} t_i^j}, \quad |t_i| < 1, \quad i = 1, \ldots, m.$$

Proof: It can be shown by induction on $n_1, \ldots, n_m$ that $0 < x_{ij} < 1$ $(1 \le i \le m$ and $1 \le j \le k)$ and $\sum_i \sum_j x_{ij} < 1$ imply $0 \le H_{\underline{n}}^{(k)} \le 1$, which shows the convergence of $g_k(t_1, \ldots, t_m; \underline{x}_1, \ldots, \underline{x}_m)$ for at least $|t_i| < 1$, since for these t_i

$$g_k(t_1, \ldots, t_m; \underline{x}_1, \ldots, \underline{x}_m) \le \sum_{n_1=1}^{\infty} \cdots \sum_{n_m=1}^{\infty} t_1^{n_1} \ldots t_m^{n_m} = \prod_i t_i (1 - t_i)^{-1}.$$

Next, using Definition 2.1, we have

$$\begin{aligned} &g_k(t_1, \ldots, t_m; \underline{x}_1, \ldots, \underline{x}_m) \\ &= t_1 \ldots t_m + \sum_{\substack{n_1=1 \\ n_1+\cdots+n_m \ge m+1}}^{\infty} \cdots \sum_{n_m=1}^{\infty} t_1^{n_1} \ldots t_m^{n_m} H_{\underline{n}}^{(k)} \\ &= t_1 \ldots t_m + \sum_{\substack{n_1=1 \\ n_1+\cdots+n_m \ge m+1}}^{\infty} \cdots \sum_{n_m=1}^{\infty} t_1^{n_1} \ldots t_m^{n_m} \sum_i \sum_j x_{ij} H_{\underline{n}-\underline{j}_i}^{(k)} \\ &= t_1 \ldots t_m + \sum_i \sum_j x_{ij} \sum_{n_1=1}^{\infty} \cdots \sum_{n_m=1}^{\infty} t_1^{n_1} \ldots t_i^{n_i+j} \ldots t_m^{n_m} H_{\underline{n}}^{(k)} \\ &= t_1 \ldots t_m + \sum_i \sum_j x_{ij} t_i^j g_k(t_1, \ldots, t_m; \underline{x}_1, \ldots, \underline{x}_m), \end{aligned}$$

from which the lemma follows.

Now let $H^{(k)}_{\underline{n},r}(\underline{x}_1, \ldots, \underline{x}_m)$ be the $(r-1)$-fold convolution of the sequence $H^{(k)}_{\underline{n}}(\underline{x}_1, \ldots, \underline{x}_m)$ with itself, i.e., $H^{(k)}_{\underline{n},r} = 0$ if some $n_i \le 0$ $(1 \le i \le m)$, and for $n_i \ge 1$ $(1 \le i \le m)$

$$(2.1) \qquad H^{(k)}_{\underline{n},r} = \begin{cases} H^{(k)}_{\underline{n}}, & \text{if } r = 1, \\ \displaystyle\sum_{c_1=1}^{n_1} \cdots \sum_{c_m=1}^{n_m} H^{(k)}_{\underline{c},r-1} H^{(k)}_{\underline{n}+\underline{1}-\underline{c}}, & \text{if } r \ge 2, \end{cases}$$

where $\underline{c} = (c_1, \ldots, c_m)$.

As a consequence of (2.1) and in view of Lemma 2.1, we have

$$(2.2) \qquad \sum_{n_1=0}^{\infty} \cdots \sum_{n_m=0}^{\infty} t_1^{n_1} \cdots t_m^{n_m} H^{(k)}_{\underline{n}+\underline{1},r}(\underline{x}_1, \ldots, \underline{x}_m) = \left(1 - \textstyle\sum_i \sum_j x_{ij} t_i^j\right)^{-r}.$$

Expanding (2.2) about $t_1 = \cdots = t_m = 0$ and using procedures similar to those of [5] and [8], we readily find the following closed formula for $H^{(k)}_{\underline{n},r}$, in terms of the multinomial coefficients.

Theorem 2.1: Let $H^{(k)}_{\underline{n},r}(\underline{x}_1, \ldots, \underline{x}_m)$ be the $(r-1)$-fold convolution of the sequence $H^{(k)}_{\underline{n}}(\underline{x}_1, \ldots, \underline{x}_m)$ with itself. Then

$$H^{(k)}_{\underline{n}+\underline{1},r}(\underline{x}_1, \ldots, \underline{x}_m) = \sum_{\sum_j j n_{ij} = n_i} \binom{n_{11} + \cdots + n_{mk} + r - 1}{n_{11}, \ldots, n_{mk}, r-1} \prod_i \prod_j x_{ij}^{n_{ij}},$$
$$n_i = 0, 1, \ldots \quad (1 \le i \le m).$$

Proof: Let $|t_i| < 1$ $(1 \le i \le m)$, $0 < x_{ij} < 1$ $(1 \le i \le m$ and $1 \le j \le k)$, and let $\sum_i \sum_j x_{ij} < 1$. Then

$$\sum_{n_1=0}^{\infty} \cdots \sum_{n_m=0}^{\infty} t_1^{n_1} \cdots t_m^{n_m} H_{\underline{n}+\underline{1},r}(\underline{x}_1, \ldots, \underline{x}_m)$$
$$= \left(1 - \textstyle\sum_i \sum_j x_{ij} t_i^j\right)^{-r}, \quad \text{by (2.2)},$$
$$= \sum_{n=0}^{\infty} \binom{n+r-1}{n} \left(\textstyle\sum_i \sum_j x_{ij} t_i^j\right)^n, \quad \text{since } \left|\textstyle\sum_i \sum_j x_{ij} t_i^j\right| < 1,$$
$$= \sum_{n=0}^{\infty} \binom{n+r-1}{n} \sum_{\sum_i \sum_j n_{ij} = n} \binom{n}{n_{11}, \ldots, n_{mk}} \prod_i \prod_j (x_{ij} t_i^j)^{n_{ij}},$$
by the multinomial theorem,
$$= \sum_{n_1=0}^{\infty} \cdots \sum_{n_m=0}^{\infty} \sum_{\substack{\sum_j n_{ij} = n_i \\ i=1,\ldots,m}} \binom{n_{11} + \cdots + n_{mk} + r - 1}{n_{11}, \ldots, n_{mk}, r-1} \prod_i \prod_j (x_{ij} t_i^j)^{n_{ij}}$$
$$= \sum_{n_1=0}^{\infty} \cdots \sum_{n_m=0}^{\infty} t_1^{n_1} \cdots t_m^{n_m} \sum_{\substack{\sum_j j n_{ij} = n_i \\ i=1,\ldots,m}} \binom{n_{11} + \cdots + n_{mk} + r - 1}{n_{11}, \ldots, n_{mk}, r-1} \prod_i \prod_j x_{ij}^{n_{ij}},$$

by replacing n_i by $n_i - \sum_j (j-1) n_{ij}$ $(1 \le i \le m)$. The theorem follows.

We proceed next to show that $H^{(k)}_{\underline{n},r}$ satisfies the following linear recurrence with variable coefficients, using procedures similar to those of [4] and [6].

Theorem 2.2: Let $H^{(k)}_{\underline{n},r}(\underline{x}_1, \ldots, \underline{x}_m)$ be the $(r-1)$-fold convolution of the sequence $H^{(k)}_{\underline{n}}(\underline{x}_1, \ldots, \underline{x}_m)$ with itself. Then

$$H^{(k)}_{\underline{n},r} = 0, \quad \text{if some } n_i \le 0 \ (1 \le i \le m), \quad H^{(k)}_{\underline{1},r} = 1,$$

and

$$H^{(k)}_{\underline{n}+\underline{1},r} = \textstyle\sum_i \sum_j x_{ij} H^{(k)}_{\underline{n}+\underline{1}-\underline{j}_i,r} + \dfrac{r-1}{n_s} \textstyle\sum_j j x_{sj} H^{(k)}_{\underline{n}+\underline{1}-\underline{j}_s,r},$$

if $n_i \ge 0$ and some $n_s \ge 1$ $(1 \le i, s \le m)$.

Proof: From the definition of $H^{(k)}_{\underline{n},r}$, we have

(2.3) $\quad H^{(k)}_{\underline{n},r} = 0$, if some $n_i \le 0$ $(1 \le i \le m)$ and $H^{(k)}_{\underline{1},r} = 1$.

Now, using (2.2) twice, we have

(2.4) $\quad H^{(k)}_{\underline{n}+\underline{1},r} = H^{(k)}_{\underline{n}+\underline{1},r+1} - \sum_i \sum_j x_{ij} H^{(k)}_{\underline{n}+\underline{1}-\underline{j}_i,r+1}$, $n_i \ge 0$ $(1 \le i \le m)$,

since the generating function of the right-hand side reduces to that of $H^{(k)}_{\underline{n}+\underline{1},r}$ Next, differentiating both sides of (2.2) with respect to t_s $(1 \le s \le m)$, we get

(2.5) $\quad n_s H^{(k)}_{\underline{n}+\underline{1},r} = r\sum_j jx_{sj} H^{(k)}_{\underline{n}+\underline{1}-\underline{j}_s,r+1}$, $n_i \ge 0$ and $n_s \ge 1$ $(1 \le i \ne s \le m)$.

Combining (2.4) and (2.5), we obtain

$$H^{(k)}_{\underline{n}+\underline{1},r} = \sum_i \sum_j x_{ij} H^{(k)}_{\underline{n}+\underline{1}-\underline{j}_i,r} + \frac{r-1}{n_s} \sum_j jx_{sj} H^{(k)}_{\underline{n}+\underline{1}-\underline{j}_s,r},$$

if $n_i \ge 0$ and some $n_s \ge 1$ $(1 \le i, s \le m)$,

by means of (2.1), which along with (2.3) establishes the theorem.

Remark 2.1: For $m = 1$, $n_1 = n$, and $\underline{x}_1 = \underline{x} = (x_1, \ldots, x_k)$, Theorems 2.1 and 2.2 reduce to the main results of Philippou and Antzoulakos [4] on multivariate Fibonacci polynomials of order k (see Theorems 2.2 and 2.3), namely,

(2.6) $\quad H^{(k)}_{n+1,r}(\underline{x}) = \sum_{\sum_j jn_j = n} \binom{n_1 + \cdots + n_k + r - 1}{n_1, \ldots, n_k, r-1} \prod_j x_j^{n_j}$, $n \ge 0$,

and

(2.7) $\quad H^{(k)}_{n+1,r}(\underline{x}) = \sum_j \frac{x_j}{n}[n + j(r-1)] H^{(k)}_{n+1-j,r}(\underline{x})$, $n \ge 1$.

Remark 2.2: For $m = 1$, $n_1 = n$, and $\underline{x}_1 = (x, \ldots, x)$, Theorems 2.1 and 2.2 reduce to Theorems 2.1(a) and 2.2 of Philippou and Georghiou [6], respectively, since for these values

$$H^{(k)}_{n_1,r}(\underline{x}_1) = F^{(k)}_{n,r}(x),$$

where $F^{(k)}_{n,r}(x)$ denotes the $(r-1)$-fold convolution of the sequence of Fibonacci-type polynomials of order k with itself.

We note in ending this section that the sequence $F^{(k)}_{\underline{n}}$ defined by

$$F^{(k)}_{\underline{n}} = \begin{cases} 0, & \text{if some } n_i \le 0 \ (1 \le i \le m), \\ 1, & \text{if } \underline{n} = \underline{1}, \\ \sum_i \sum_j F^{(k)}_{\underline{n}-\underline{j}_i}, & \text{elsewhere,} \end{cases}$$

may be called the multiple Fibonacci sequence of order k, since for $m = 1$ and $n_1 = n$ (≥ 0) it reduces to $F^{(k)}_n$, the Fibonacci sequence of order k (see, e.g., Philippou and Muwafi [7]). It may be noted that

(2.8) $\quad F^{(k)}_{\underline{n}+\underline{1}} = \sum_{\sum_j jn_{ij} = n_i} \binom{n_{11} + \cdots + n_{mk}}{n_{11}, \ldots, n_{mk}}$, $n_i = 0, 1, \ldots$ $(1 \le i \le m)$.

which follows from Theorem 2.1 for $r = 1$ and $x_{ij} = 1$ $(1 \le i \le m$ and $1 \le j \le k)$.

3. Recurrence Relations for the Multivariate Negative Binomial Distributions of Order k

In this section, we employ Theorems 2.1 and 2.2 to derive a recurrence relation for calculating the probabilities of the following multivariate negative binomial distribution of order k of Philippou, Antzoulakos, and Tripsiannis [8].

Definition 3.1: A random vector $\underline{N} = (N_1, \ldots, N_m)$ is said to have the multivariate negative binomial distribution of order k with parameters r, $q_{11}, \ldots, q_{mk}$ ($r > 0$, $0 < q < 1$ for $1 \le i \le m$ and $1 \le j \le k$, and $q_{11} + \cdots + q_{ij} < 1$), to be denoted by $\text{MNB}_k(r;\, q_{11}, \ldots, q_{mk})$, if

$$P(N_1 = n_1, \ldots, N_m = n_m)$$
$$= p^r \sum_{\sum_j j n_{ij} = n_i} \binom{n_{11} + \cdots + n_{mk} + r - 1}{n_{11}, \ldots, n_{mk}, r - 1} \Pi_i \Pi_j q_{ij}^{n_{ij}}, \quad n_i = 0, 1, \ldots \; (1 \le i \le m),$$

where $p = 1 - q_{11} - \cdots - q_{mk}$.

Analogous recurrences are also given for the type I, type II, and extended multivariate negative binomial distributions of order k of [8], denoted by

$$\overline{\text{MNB}}_{k,\,\text{I}}(r;\, Q_1, \ldots, Q_m),\ \text{MNB}_{k,\,\text{II}}(r;\, Q_1, \ldots, Q_m),\ \text{and}$$
$$\overline{\text{MENB}}_k(r;\, q_{11}, \ldots, q_{mk}).$$

These distributions result by applying to the parameters of $\text{MNB}_k(r;\, q_{11}, \ldots, q_{mk})$ the following transformations, respectively:

(a) $q_{ij} = P^{j-1}Q_i$ ($0 < Q_i < 1$ for $1 \le i \le m$, $\sum_i Q_i < 1$ and $P = 1 - \sum_i Q_i$);

(b) $q_{ij} = Q_i/k$ ($0 < Q_i < 1$ for $1 \le i \le m$, $\sum_i Q_i < 1$ and $P = 1 - \sum_i Q_i$);

(c) $q_{ij} = P_1 P_2 \ldots P_{j-1} Q_{ij}$ ($P_0 = 1$, $0 < Q_{ij} < 1$ for $1 \le i \le m$ and $1 \le j \le k$, $\sum_i Q_{ij} < 1$ and $P_j = 1 - \sum_i Q_{ij}$ for $1 \le j \le k$).

We note first the following proposition that relates the multivariate negative binomial distribution of order k to the generalized multivariate Fibonacci polynomials of the same order.

Proposition 3.1: Let $\underline{N} = (N_1, \ldots, N_m)$ be a random vector distributed as

$$\text{MNB}_k(r;\, q_{11}, \ldots, q_{mk}),$$

and let $H^{(k)}_{\underline{n},\,r}$ be the $(r - 1)$-fold convolution of the sequence $H^{(k)}_{\underline{n}}$ with itself. Then

$$P(N_1 = n_1, \ldots, N_m = n_m) = p^r H^{(k)}_{\underline{n}+\underline{1},\,r}(\underline{q}_1, \ldots, \underline{q}_m), \quad n_i = 0, 1, \ldots, 1 \le i \le m,$$

where $\underline{q}_i = (q_{i1}, \ldots, q_{ik})$, $i = 1, \ldots, m$.

Proof: The proof is a direct consequence of Theorem 2.1 and Definition 3.1.

We proceed now to derive a recurrence relation for calculating the probabilities of $\text{MNB}_k(r;\, q_{11}, \ldots, q_{mk})$.

Theorem 3.1: Let $\underline{N} = (N_1, \ldots, N_m)$ be a random vector distributed as

$$\text{MNB}_k(r;\, q_{11}, \ldots, q_{mk}),$$

and set

$$P_{\underline{n},\,r} = P(N_1 = n_1, \ldots, N_m = n_m).$$

Then

$$P_{\underline{n},\,r} = \begin{cases} 0, & \text{if some } n_i \le -1 \ (1 \le i \le m), \\ p^r, & \text{if } n_1 = \cdots = n_m = 0, \\ \sum_i \sum_j q_{ij} P_{\underline{n}-\underline{j}_i,\,r} + \dfrac{r-1}{n_s} \sum_j j q_{sj} P_{\underline{n}-\underline{j}_s,\,r}, & \text{if } n_i \ge 0 \text{ and some } n_s \ge 1 \ (1 \le i,\ s \le m). \end{cases}$$

Proof: If some $n_i \le -1$ $(1 \le i \le m)$, $(N_1 = n_1, \ldots, N_m = n_m) = \emptyset$, which implies $P_{\underline{n}, r} = P(\emptyset) = 0$. If $n_1 = \cdots = n_m = 0$, Definition 3.1 gives $P_{\underline{n}, r} = p^r$. If $n_i \ge 0$ and some $n_s \ge 1$ $(1 \le i, s \le m)$, we have

$$\begin{aligned} P_{\underline{n}, r} &= p^r H^{(k)}_{\underline{n}+\underline{1}, r}(\underline{q}_1, \ldots, \underline{q}_m), \text{ by Proposition 3.1,} \\ &= p^r\Big\{\textstyle\sum_i \sum_j q_{ij} H^{(k)}_{\underline{n}+\underline{1}-\underline{j}_i, r}(\underline{q}_1, \ldots, \underline{q}_m) \\ &\quad + \dfrac{r-1}{n_s} \textstyle\sum_j j q_{sj} H^{(k)}_{\underline{n}+\underline{1}-\underline{j}_s, r}(\underline{q}_1, \ldots, \underline{q}_m)\Big\}, \text{ by Theorem 2.2,} \\ &= \textstyle\sum_i \sum_j q_{ij} P_{\underline{n}-\underline{j}_i, r} + \dfrac{r-1}{n_s} \textstyle\sum_j j q_{sj} P_{\underline{n}-\underline{j}_s, r}, \text{ by Proposition 3.1.} \end{aligned}$$

Using the transformations (a), (b), and (c), respectively, Theorem 3.1 now reduces to the following corollaries.

Corollary 3.1: Let $\underline{N} = (N_1, \ldots, N_m)$ be a random vector distributed as

$$\overline{\text{MNB}}_{k,\,\text{I}}(r;\, q_1, \ldots, q_m),$$

and set

$$P_{\underline{n}, r} = P(N_1 = n_1, \ldots, N_m = n_m).$$

Then

$$P_{\underline{n}, r} = \begin{cases} 0, & \text{if some } n_i \le -1 \ (1 \le i \le m), \\ p^{kr}, & \text{if } n_1 = \cdots = n_m = 0, \\ \sum_i \sum_j p^{j-1} q_i P_{\underline{n}-\underline{j}_i, r} + \dfrac{r-1}{n_s} \sum_j j p^{j-1} q_s P_{\underline{n}-\underline{j}_s, r}, & \\ \qquad\qquad \text{if } n_i \ge 0 \text{ and some } n_s \ge 1 \ (1 \le i, s \le m). \end{cases}$$

Corollary 3.2: Let $\underline{N} = (N_1, \ldots, N_m)$ be a random vector distributed as

$$\text{MNB}_{k,\,\text{II}}(r;\, q_1, \ldots, q_m),$$

and set

$$P_{\underline{n}, r} = P(N_1 = n_1, \ldots, N_m = n_m).$$

Then

$$P_{\underline{n}, r} = \begin{cases} 0, & \text{if some } n_i \le -1 \ (1 \le i \le m), \\ p^{r}, & \text{if } n_1 = \cdots = n_m = 0, \\ \sum_i \sum_j \dfrac{q_i}{k} P_{\underline{n}-\underline{j}_i, r} + \dfrac{r-1}{n_s} \sum_j j \dfrac{q_s}{k} P_{\underline{n}-\underline{j}_s, r}, & \\ \qquad\qquad \text{if } n_i \ge 0 \text{ and some } n_s \ge 1 \ (1 \le i, s \le m). \end{cases}$$

Corollary 3.3: Let $\underline{N} = (N_1, \ldots, N_m)$ be a random vector distributed as

$$\overline{\text{MENB}}_k(r;\, q_{11}, \ldots, q_{mk}),$$

and set

$$P_{\underline{n}, r} = P(N_1 = n_1, \ldots, N_m = n_m).$$

Then

$$P_{\underline{n}, r} = \begin{cases} 0, & \text{if some } n_i \le -1 \ (1 \le i \le m), \\ (p_1 \cdots p_k)^{r}, & \text{if } n_1 = \cdots = n_m = 0, \\ \sum_i \sum_j p_1 \cdots p_{j-1} q_{ij} P_{\underline{n}-\underline{j}_i, r} + \dfrac{r-1}{n_s} \sum_j p_1 \cdots p_{j-1} q_{sj} P_{\underline{n}-\underline{j}_s, r}, & \\ \qquad\qquad \text{if } n_i \ge 0 \text{ and some } n_s \ge 1 \ (1 \le i, s \le m). \end{cases}$$

For $m = 1$, Theorem 3.1 and Corollaries 3.1-3.3 reduce to known recurrences concerning respective univariate negative binomial distributions of order k (see [4] and [6]). For $k = 1$, Theorem 3.1 (or any one of Corollaries 3.1-3.3) provides the following recurrence for the probabilities of $\text{MNB}(r;\, q_1, \ldots, q_m)$, the usual multivariate negative binomial distribution,

$$P_{\underline{n},r} = \begin{cases} 0, & \text{if some } n_i \le -1 \ (1 \le i \le m), \\ p^r, & \text{if } n_1 = \cdots = n_m = 0, \\ \sum_i q_i P_{\underline{n}-\underline{1}_i, r} + \dfrac{r-1}{n_s} q_s P_{\underline{n}-\underline{1}_s, r}, & \text{if } n_i \ge 0 \text{ and some } n_s \ge 1 \ (1 \le i, s \le m), \end{cases}$$

which does not seem to have been noticed before.

Remark 3.1: For $r = 1$, Theorem 3.1 and Corollaries 3.1-3.3 provide recurrences for the probabilities of respective multivariate geometric distributions of order k of [8], defined by

$$MG_k(q_{11}, \ldots, q_{mk}) = MNB_k(1; q_{11}, \ldots, q_{mk}),$$
$$\overline{MG}_{k,\mathrm{I}}(q_1, \ldots, q_m) = \overline{MNB}_{k,\mathrm{I}}(1; q_1, \ldots, q_m),$$
$$MG_{k,\mathrm{II}}(q_1, \ldots, q_m) = MNB_{k,\mathrm{II}}(1; q_1, \ldots, q_m),$$

and
$$\overline{MEG}_k(q_{11}, \ldots, q_{mk}) = \overline{MENB}\ (1; q_{11}, \ldots, q_{mk}).$$

The resulting recurrence for $\overline{MEG}_k(q_{11}, \ldots, q_{mk})$ has also been obtained in [5], via a different method.

We note in ending this paper that another derivation of Theorem 3.1, without employing the sequence of generalized multivariate Fibonacci polynomials of order k, has been obtained by Antzoulakos and Philippou (see [2]).

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