SOME RESULTS CONCERNING POLYOMINOES

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INTRODUCTORY REMARKS

An n-omino is a plane figure composed of n connected unit squares joined edge on edge. In the early nineteen hundreds, Henry Dudeney, the famous British puzzle expert, and the Fairy Chess Review popularized problems involving n-ominoes which they represented as figures cut from checkerboards. Solomon Golomb seems to have been the first mathematician to treat the subject seriously when as a graduate student at Harvard in 1954, he published "Checkerboards and Polyominoes" in the American Mathematical Monthly. Since 1954, several articles have appeared (see References); in particular, R. C. Read [9] and Murray Eden [2] have discussed the problem of finding or estimating the number p(n) of n-ominoes for a given n. From their results it is now known that for large n

$$c_1^n < p(n) < c_2^n$$

where c_1 and c_2 are certain positive constants greater than 1. In the first part of this paper we enumerate a subset of n-ominoes and provide an improved lower bound for p(n); later we discuss other problems of this sort and conclude with a brief exposition of problems dealing with configurations of n-ominoes.

MOSER'S BOARD PILE PROBLEM

In the following it will be convenient to have certain conventions. We say the region between y = n-1 and y = n is the n^{th} row and call a rectangle of width one a strip. The first square on the left in a strip located in a row is called the initial square of the strip; an n-omino is located in the plane when some square in the n-omino exactly covers a square in the plane lattice. The set of all incongruent n-ominoes will be denoted by P(n) and for convenience we think of the elements of P(n) located in arbitrary regions of the plane. Ignoring changes in position due to translations, each element of P(n) has eight or less positions with respect to 90° rotations about the origin and reflections along the x or y axes; taking two n-ominoes to be distinct if one cannot be translated to cover the other, we find a new set S(n) from P(n) by including rotations and reflections of n-ominoes in P(n) in S(n).

The problem which is now to be discussed was probably first posed by Leo Moser in private correspondence with the present author; later he posed it in a different form at the 1963 Number Theory Conference held at the University of Colorado. Eden [2] also discusses the problem, but his results are not as complete as those given here. The problem is to enumerate a subset B(n) of S(n) which contains n-ominoes having the property that they can be translated in such a way that they are entirely in the first and second quadrants with exactly one strip in the first row with its initial square at the origin and each row after the first has no more than one strip in it. Such n-ominoes may be visualized as side elevations of board piles consisting of boards of various lengths which generally have not been stacked carefully, see Figure 1.

Moser noted that if b(n) denotes the number of elements in B(n), then

$$(1) \qquad b(n) = \sum (a_1 + a_2 - 1)(a_2 + a_3 - 1) \ldots (a_{i-1} + a_i - 1)$$

where the summation extends over all compositions $a_1 + a_2 \ldots + a_i = n$ of n. The relation in (1) can be established by the following combinatorial argument. For each composition $a_1 + a_2 + \ldots + a_i$ of n there is a subset of B(n) consisting of n-ominoes which have a strip of a_t squares in the t^{th} row $(t = 1, 2, \ldots, i)$; the number of n-ominoes in each of these subsets is 1 if $i = 1$ which corresponds to the value of the empty product in the sum (in this there is a strip n units long in the first row) and $(a_1 + a_2 - 1)(a_2 + a_3 - 1) \ldots (a_{i-1} + a_i - 1)$ if $i \geq 2$. This follows since there are exactly $(a_{t-1} + a_t - 1)$ ways to join the strip of a_t squares in the t^{th} row to the strip of a_{t-1} squares in the row below and the total number of ways to connect up the strips to form an n-omino would be the product of all of these alternatives. The subsets corresponding to the compositions of n are exhaustive and disjoint in $B(n)$, so that $b(n)$ is the sum of the number of elements in each subset, which is (1).

The relation for $b(n)$ given by (1) does not furnish a very handy device for computing $b(n)$, but as Eden has shown it is helpful in estimating $b(n)$. Rather than attempt to sum (1) by purely algebraic manipulations, we retain the geometric interpretation of the problem so that combinatorial arguments can be more easily applied toward finding a recursion relation for $b(n)$.

To find a recursion relation for $b(n)$ we define subsets $B_r(n)$ $(r = 1, 2, \ldots, n)$ of $B(n)$ which contain n-ominoes with a strip of exactly r squares in the first row and let $b_r(n)$ denote the number of elements in $B_r(n)$. It is obvious that the subsets $B_r(n)$ $(r = 1, 2, \ldots, n)$ are exhaustive and disjoint in $B(n)$ so that we have immediately

$$b(n) = \sum_{r=1}^{n} b_r(n) \quad . \tag{2}$$

By definition of $B_n(n)$, $b_n(n) = 1$. Consider the elements of $B_r(n)$ with $r < n$; each element of $B_r(n)$ consists of a strip of r squares in the first row with some element of $B(n-r)$ located in the rows above the first. The situation can be appraised more concisely when one considers the number of ways an element of the subset $B_i(n-r)$ of $B(n-r)$ can be attached to the strip of r squares in the first row so that the n-ominoes formed will be an element of $B_r(n)$. Clearly this can be done in $r + i - 1$ ways, so that exactly $(r + i - 1)\, b_i(n-r)$ of the elements of $B_r(n)$ have an element of $B_i(n-r)$ connected to the strip of r squares in the first row. Since the subsets $B_i(n-r)$ $(i = 1, 2, \ldots, n-r)$ of $B(n-r)$ are exhaustive, disjoint subsets, it follows that

$$b_r(n) = \sum_{i=1}^{n-r} (r + i - 1)\, b_i(n-r) \quad \text{for } r < n \quad . \tag{3}$$

It will be seen presently that the relations in (2) and (3) are enough to find the desired recursion relation for $b(n)$. Before this result can be given, we have to prove a few lemmas.

Lemma 1: If $n > 1$, $b_r(n) - b_{r-1}(n-1) = b(n-r)$.

Proof: Using (3) it is seen that

$$b_r(n) - b_{r-1}(n-1) = \sum_{i=1}^{n-r} (r+i-1)b_i(n-r) - \sum_{i=1}^{n-r} (r+i-2)b_i(n-r)$$

$$= \sum_{i=1}^{n-r} b_i(n-r) \quad ,$$

but according to (2), the last expression is precisely $b(n-r)$, so the proof is finished.

Lemma 2: If $n > 1$, $b(n) = 2\,b(n-1) + b_1(n) - b_1(n-1)$.

Proof: Using relations for $b(n)$ and $b(n-1)$ given by (2), it is seen that

$$(5) \qquad b(n) - b(n-1) = \sum_{i=1}^{n} b_i(n) - \sum_{i=1}^{n-1} b_i(n-1)$$

$$= b_1(n) + \sum_{i=2}^{n-1} \left\{ b_i(n) - b_{i-1}(n-1) \right\} \quad ;$$

but according to Lemma 1, $b(n-i)$ can be substituted for $b_i(n) - b_{i-1}(n-1)$ in the last member of (5) so that making this substitution and transposing $-b(n-1)$ from the first to the last member gives

$$(6) \qquad b(n) = b_1(n) + \sum_{i=1}^{n-1} b(n-i) \quad .$$

Now using relations given by (6) for $b(n)$ and $b(n-1)$ we have

$$(7) \quad b(n) - b(n-1) = b_1(n) + \sum_{i=1}^{n-1} b(n-i) - b_1(n-1) - \sum_{i=1}^{n-2} b(n-1-i)$$

$$= b_1(n) - b_1(n-1) + b(n-1);$$

the desired result is obtained by adding $b(n-1)$ to the first and last members of (7).

Lemma 3: $b_1(n) = 4\, b_1(n-1) - 4\, b_1(n-2) + b_1(n-3) + 2\, b(n-3)$.

Proof: Taking $r = 1$ in (3) gives an expression for $b_1(n)$; namely,

$$b_1(n) = \sum_{i=1}^{n-1} i\, b_i(n-1) \quad . \tag{8}$$

Using relations for $b_1(n)$ and $b_1(n-1)$ given by (8) and substituting $b(n-2-i)$ for $b_{i+1}(n-1) - b_i(n-2)$ and $b(n-1)$ for

$$\sum_{i=1}^{n-1} b_i(n-1)$$

when they occur, it is seen that

$$\begin{aligned} b_1(n) - b_1(n-1) &= \sum_{i=1}^{n-1} i\, b_i(n-1) - \sum_{i=1}^{n-2} i\, b_i(n-2) \\ &= \sum_{i=1}^{n-1} b_i(n-1) + \sum_{i=1}^{n-2} i\, b_{i+1}(n-1) - \sum_{i=1}^{n-2} i\, b_i(n-2) \\ &= b(n-1) + \sum_{i=1}^{n-2} i \left\{ b_{i+1}(n-1) - b_i(n-2) \right\} \\ &= b(n-1) + \sum_{i=1}^{n-2} i\, b(n-2-i) \quad . \end{aligned} \tag{9}$$

Adding $b_1(n-1)$ to each member of the equality and dropping the last term in the sum in the right member of (9) (since $b(0) = 0$) a new relation for $b_1(n)$ is obtained:

$$b_1(n) = b_1(n-1) + b(n-1) + \sum_{i=1}^{n-3} (n-2-i)\, b(i) \quad . \tag{10}$$

This time using expressions for $b_1(n)$ and $b_1(n-1)$ given by (10) and again writing a relation for $b_1(n) - b_1(n-1)$, one obtains after a few algebraic manipulations

$$(11) \qquad b_1(n) = 2b_1(n-1) - b_1(n-2) - 2b(n-2) + \sum_{i=1}^{n-1} b(i) .$$

Repeating the same procedure as before only this time using expressions for $b_1(n)$ and $b_1(n-1)$ given by (11) yields

$$(12) \quad b_1(n) = 3b_1(n-1) - 3b_1(n-2) + b_1(n-3) + b(n-1) - 2b(n-2) + 2b(n-3);$$

but by Lemma 2, $b(n-1) - 2b(n-2) = b_1(n-1) - b_1(n-2)$ so that substituting the latter quantity for the former in (12) gives the desired result.

Theorem 1: $b(1) = 1$, $b(2) = 2$, $b(3) = 6$, $b(4) = 19$, and
$b(n) = 5b(n-1) - 7b(n-2) + 4b(n-3)$ for $n > 4$.

Proof: The values of $b(i)$ $(i = 1, 2, 3, 4)$ can be computed directly from (1) or by taking $b(1) = b_1(1) = 1$ the relations in (2) and (3) can be used together for the same purpose. Lemmas 2 and 3 provide the linear difference equations involving $b_1(n)$ and $b(n)$ which can be used to find

$$(13) \qquad b(n) = 5b(n-1) - 7b(n-2) + 4b(n-3) ,$$

$$(14) \qquad b_1(n) = 6b_1(n-1) - 12b_1(n-2) + 11b(n-3) - 4b(n-4),$$

which completes the proof.

The auxiliary equation for (13) has one real root greater than 3.2 so that for n sufficiently large

$$(15) \qquad b(n) > (3.2)^n .$$

We conclude from earlier remarks that $B(n)$ contains at least $b(n)/8$ incongruent n-ominoes, so that we can also replace $b(n)$ in (15) with $p(n)$.

Having disposed of the more difficult problem first, we now turn attention to solving an easier and related problem which was posed and solved by Moser.

Let $C(n)$ be the subset of $B(n)$ which contains all n-ominoes having the property that the initial square of the strip in the k^{th} row is no further to the left than the initial square of the strip in the $(k-1)^{st}$ row. Recall from the definition of $B(n)$ that the initial square of the strip in the first row is always located at the origin. Using a combinatorial argument similar to the one provided for the proof of (1), it is easy to prove

$$c(n) = \sum_{a_1+a_2+\ldots+a_i=n} a_1 a_2 \ldots a_{i-1} , \tag{16}$$

where $c(n)$ denotes the number of elements in $C(n)$. Applying the methods he gave in [8] , Moser was able to show from (16):

Theorem 2: $c(n)$ is equal to the $(2n-1)^{st}$ Fibonacci number.

We will give an alternate proof using the same idea used in the proof of Theorem 1. Let $C_i(n)$ be the subset of $C(n)$ which contains all n-ominoes having strips of exactly i squares in the first row. Clearly the subsets $C_i(n)$ $(i = 1, 2, \ldots, n)$ are exhaustive and disjoint in $C(n)$ so that letting $c_i(n)$ denote the number of elements in $C_i(n)$ we have

$$c(n) = \sum_{i=1}^{n} c_i(n) . \tag{17}$$

Next, it is easy to see that $c_n(n) = 1$, and for $i < n$, $c_i(n) = i\,c(n-i)$ since each element of $C(n-i)$ can be joined exactly i ways to the strip of i squares in the first row so as to form an element of $C_i(n)$; the n-ominoes thus formed obviously comprise all the elements of $C_i(n)$. Substituting the expressions just found for $c_i(n)$ into (17) we obtain

$$c(n) = 1 + \sum_{i=1}^{n-1} i\,c(n-i) . \tag{18}$$

Using expressions for $c(n)$ and $c(n-1)$ given by (18) we can combine the sums in $c(n) - c(n-1)$ to find

$$c(n) - c(n-1) = \sum_{i=1}^{n-1} c(i) ,$$

or

$$c(n) = c(n-1) + \sum_{i=1}^{n-1} c(i) .$$

Now using expressions for $c(n)$ and $c(n-1)$ given by (19) we can combine the sums in $c(n) - c(n-1)$ and deduce

$$c(n) = 3\,c(n-1) - c(n-2). \tag{20}$$

It is easy to prove that the Fibonacci numbers with odd indices satisfy the recurrence relation in (20). Also, using (16) we find $c(1) = f_1$ and $c(2) = f_3$ (f_i denotes the i^{th} Fibonacci number as usual) so that the sequences $\{c_i\}$ and $\{f_{2i-1}\}$ must be identical. Editorial Note: See H-50 Dec. 1964 and note notational differences.

N-OMINOES ENCLOSED IN RECTANGLES

R. C. Read [9] has treated the problem of enumerating the n-ominoes which "fit" into a $p \times q$ rectangle. An n-omino is said to fit in a $p \times q$ rectangle if it is the smallest rectangle in which the n-omino can be drawn with the sides of its squares parallel to the sides of the rectangle. Read's methods give exact counts of the n-ominoes in the sets considered; however, it is possible to obtain lower bounds for these numbers with less effort using similar ideas. To illustrate we will consider the problem of estimating from below the number $s_2(n)$ of n-ominoes which fit in a $2 \times k$ rectangle; we call this set of n-ominoes $S_2(n)$. Two elements are distinct if they are incongruent, so $S_2(n)$ is a subset of $P(n)$.

First, we observe that each element of $S_2(n)$ can be located entirely in the first quadrant in rows 1 and 2 with a square located at the origin. If each element of $S_2(n)$ is situated in the way just described in every way possible, a new set $U(n)$ is obtained where two elements are distinct if one does not exactly cover the other. Clearly, $u(n)$, the number of elements in $U(n)$, is less than or equal to $4s_2(n)$. Now $U(n)$ can be divided into two sets $U''(n)$ and $U'(n)$ consisting respectively of n-ominoes having and not having a square in the second row attached to the square at the origin. Let the number of elements in $U'(n)$ and $U''(n)$ be $u'(n)$ and $u''(n)$ respectively. Now it is easy to see that

$$u'(n) = u'(n-1) + u''(n-1) \tag{21}$$

since every element of $U''(n-1)$ and $U'(n-1)$ can be translated a unit to the right of the origin and a square located at the origin to give an element of $U'(n)$ and every element is obviously obtained in this fashion. It is also easy to prove

$$u''(n) = 2u'(n-2) + u''(n-2) \tag{22}$$

since every element of $U''(n-2)$ and every element of $U'(n-2)$ and its horizontal reflection can be translated a unit to the right of the origin and two squares added (one at the origin, the other attached above it) to form every element of $U''(n)$.

Using (21) and (22) we can find

$$u'(n) = u'(n-1) + u'(n-2) + u'(n-3) \tag{23}$$

and

$$u''(n) = u''(n-1) + u''(n-2) + u''(n-3), \tag{24}$$

so that it becomes evident from $u(n) = u'(n) + u''(n)$ that

$$u(n) = u(n-1) + u(n-2) + u(n-3) \quad . \tag{25}$$

Since $u(n)/4 \le s_2(n)$, (25) provides a relation for estimating $s_2(n)$. The same procedure can be used for estimating the number of elements in $S_k(n)$ consisting of n-ominoes which fit in $k \times q$ rectangles.

N-OMINO CONFIGURATIONS

Problems involving n-omino configurations have enjoyed a great popularity among mathematical recreationists [4], [6]. We plan to devote a small amount of space to giving an exposition of problems which may be of interest to the mathematician. Generally these problems have the following form: given a region of area A and a set of n-ominoes having a combined area also A; can one cover the region with the set?

We say a set exactly covers a region when there is no overlap and no part of the region is left uncovered. It would be interesting to know necessary conditions that an n-omino be such that an unlimited number of copies could be used to exactly cover the plane. A related problem is to determine necessary conditions that some number of copies of a given n-omino could be used to exactly cover a rectangle. Thus, some easily proved necessary conditions are given by:

(i) if an n-omino has two lines of symmetry and a set of these n-ominoes exactly covers a rectangle, then the n-omino is itself a rectangle.

(ii) if an n-omino fits in a $p \times q$ rectangle and covers diagonally opposite corners of the rectangle, and a set of these n-ominoes can be used to exactly cover a rectangle, then the n-omino is itself a rectangle.

A rectangle exactly covered with a set of congruent n-ominoes is minimal when no rectangle of smaller area can be exactly covered with a set of the same n-ominoes containing fewer elements. It is easy to prove that there is an unlimited number of minimal rectangles involving either two or four n-ominoes. Figures 2, 3, 4 and 5 show instances of minimal rectangles involving more than four n-ominoes. Are there infinitely many cases of minimal rectangles which involve more than four n-ominoes (no two cases involving similar n-ominoes)? Are there minimal rectangles involving an odd number of n-ominoes which are not themselves rectangles?

Note that the configurations depicted in Figures 1, 2, 3 and 4 are symmetric with respect to the centers of the rectangles. Can this always be done in minimal rectangles?

GENERALIZATIONS OF N-OMINOES

In [5], Golomb suggests that one could try to determine or estimate the number of distinct ways n equilateral triangles or n regular hexagons could be simply connected edge on edge. Using 1, 2, 3, 4, 5 or 6 hexagons 1, 1, 3, 7, 22 or 83 combinations respectively result; so far no upper or lower bounds for the terms of this sequence have been given.

There is no reason why regular k-gons could not be used for cells in such combinatorial problems; overlapping of cells could be permitted so long as no cell exactly covered another. Thus, where at most four squares or three hexagons might have a vertex in common, at most ten pentagons might have a vertex in common. The number of distinct ways to join two regular k-gons is one; the number of ways to join three regular k-gons is the greatest integer in $k/2$. Perhaps it would not be difficult to determine in how many ways four or five regular k-gons could be joined together edge on edge so that distinct simply connected figures are formed.

Still another generalization of n-ominoes which seems not to have been considered is joining squares together edge on edge in three or more dimensions. The number of ways of joining k cubes face on face in three dimensions (including mirror images of some pieces) is 1, 1, 2, 8, 29, and 166 for $k = 1, 2, 3, 4, 5$, and 6 respectively; no bounds have been given for the terms of this sequence nor has much been done in a serious vein connected with the packing of space with these three dimensional analogues of polyominoes.

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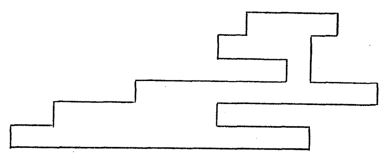

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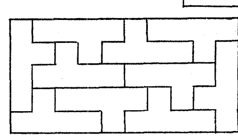

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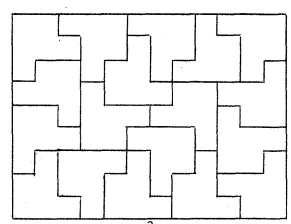

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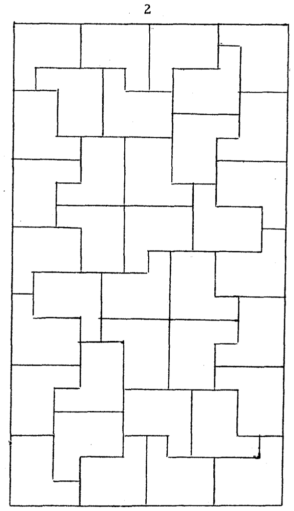

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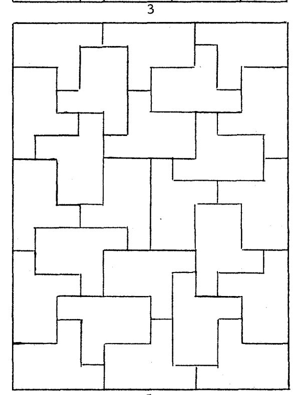

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