emerging for complex subscripts. This fine reference [2] was brought to our attention by Prof. Tyre A. Newton.

REFERENCES

1. The square root of Q was suggested by Maxey Brooke in a letter.
2. J. C. Amson, "Lucas Functions," Eureka: The Journal of the Archimedeans, (Cambridge University), No. 26, October, 1963, pp. 21-25.
3. S. L. Basin and Verner E. Hoggatt, Jr., "A Primer on the Fibonacci Sequence, Part II," Fibonacci Quarterly, 1:2, pp. 61-68.

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LETTER TO THE EDITOR

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I read with great interest your recent paper "On the Ordering of the Fibonacci Sequences." The general idea underlying your ordering procedure is excellent, but the representation can be improved and (possibly obscure) relationships may be brought to light.

Consider (for the time being) sequences for which $D \geq 11$. For reasons which will soon become clear I prefer to define the number f_o (in your notation) as the first term in the sequence, ϕ, say. You correctly pointed out that "a negative sequence may be obtained from a positive sequence by changing the signs of all terms"...; however, there is another (rather simple) operation which establishes an equivalence between two sequences. Consider a sequence

$$\ldots\ldots \phi_{-4}, \ \phi_{-3}, \ \phi_{-2}, \ \phi_{-1}, \phi_0, \phi_1, \phi_2, \phi_3, \phi_4, \ldots\ldots$$

and assume, for convenience, that the monotonic portion is positive. It is easy to verify that ϕ_{-n} is positive (negative) if n is even (odd) where n is a non-negative integer. Next view an associated sequence $\{\phi'\}$ defined by

$$\phi'_{\pm n} = \phi_{\mp n} \qquad \text{if } n \text{ is even}$$

$$\phi'_{\pm n} = \phi_{\mp n} \qquad \text{if } n \text{ is odd.}$$

It is elementary to show that $\{\phi'\}$ is a Fibonacci sequence (with the monotonic part positive) - thus Fibonacci sequences typically appear in pairs - $\{\phi''\}$ being $\{\phi\}$ - although the possibility of a sequence being self-associated (and thus appearing to be single) cannot be ruled out a priori. Now for a series to be self-associated in the following must hold - $\phi_{-1} = +\phi_{+1}$ so that the central term ϕ_o becomes

$$\phi_o = \phi_{+1} - \phi_{-1} = 2\phi_{+1}$$

and, if we are interested only in sequences which are not integral multiples of other sequences, it becomes clear that one, and only one, such sequence exists, to wit

$$\ldots\ldots, -4, 3, -1, 2, 1, 3, 4, \ldots\ldots$$

whose D equals 5. Let this sequence be denoted as the ordinary self-associated sequence. However, there exists in addition an extraordinary self-associated sequence. If we admit (which we did not before) the possibility $\phi_o = 0$, we have

$$\phi_{-1} = \phi_{+1}$$

and the only "prime" solution is the Fibonacci sequence

$$\ldots\ldots, -3, 2, -1, 1, 0, 1, 1, 2, 3, \ldots\ldots$$

You note, of course, that in this single case f_o equals ϕ_o and not ϕ_{+1}. This sequence is indeed extraordinary in several respects: In contradistinction to all other sequences it has the property that ϕ_{-n} is positive (negative) if n is odd (even). Also $\phi_o < |\phi_{\pm 1}|$ is true in this case whereas in all other cases the inequality holds in the opposite direction. An exceptional behavior of D will be discussed in this letter.

It is then my proposal to characterize the Fibonacci sequences not by (f_o, f_1) but rather by (ϕ_{-1}, ϕ_{+1}). This representation has numerous advantages: The two mutually dual Fibonacci sequences may be represented by one pair of brackets, e.g., what you represent as (1, 4) and (2, 5) would become in my notation (-2, 1) and (-1, 2); both in one representation would be written as [2, 1] with the agreement that the larger (in absolute value) number precedes the smaller number. The ordinary self-associated (or self-dual) sequence would be [1, 1] whereas the extraordinary self-dual sequence deserves special notation, e.g. (1, 1).

Consider now the quantity D as defined in your paper

$$D = f_1^2 - f_1 f_o - f_o^2$$

In terms of ϕ_{-1} and ϕ_{+1} this becomes

$$D = \phi_{-1}^2 - 3\phi_{-1}\phi_{+1} + \phi_{+1}^2 = (\phi_{+1} - \phi_{-1})^2 - \phi_{-1}\phi_{+1} \quad .$$

Again since for the original Fibonacci sequence ϕ_0 is differently defined (in terms of your f's) we get D = -1 in this case on my definition but this is not disconcerting. To my mind the original Fibonacci sequence is sufficiently extraordinary (on comparison with other such sequences) that it deserves a D with a sign different from that of the others. Inspection of the D's as presented in your paper leads me to the following conjectures (I am inclined to think that (a) it is not difficult to prove them, (b) it has been done so before - thus I have not taken the trouble).

Let prime numbers of the form $10n \pm 1$ be represented by g_k. We have then

(1) The set of feasible D's is made up of -1, 5, all g_k's, all products of g_k's which we denote by Q_m (i.e. $Q_m = g_i^a g_j^b \ldots g_k^c$), all numbers of the form $5g_k$, and all numbers of the form $5Q_m$. In other words a necessary condition for an integer to be a D is that it belongs to the set $\{-1, 5, g_k, Q_m, 5g_k, 5Q_m\}$.

(2) Each number in the above set may indeed be found in the list of D's. In other words, the above is also a sufficient condition.

(3) The number of sequences associated with a given value of D is simply related to its factorization properties. I reserve a final formulation of my conjecture on this part until I have seen more "experimental material," i.e., a table of D's (with associated sequences) between 1000 and 2000. It is already obvious that for -1 and 5 we get the self-dual sequences and for each g_k and $5g_k$ we have one pair of dual sequences. As for a Q_m it is obvious that if it equals g_k^a $(a > 1)$ we have again one associated pair, but for the case $Q_m = g_i^a g_j^b \ldots a_k^c$ the number of associated pairs is a function of the degree of "compositeness" and this should be looked into a little more carefully by means of an extended Table. Finally, the number of pairs of Fibonacci sequences associated with $5Q_m$ is identical with the number of pairs associated with Q_m.

If you are aware of literature relating to these conjectures, kindly let me know. Also if you have an extended table of the D's I should appreciate a copy.

I hope some of my remarks may have been of use for ordering and classification purposes.

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