EXPLORING SCALENE FIBONACCI POLYGONS

Proposed by BROTHER U. ALFRED on page 60, October 1963
"The Fibonacci Quarterly"
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The sequence of Fibonacci numbers may be defined by

(1) $$F_0 = 0,\ F_1 = 1,\ \text{and } F_n = F_{n-1} + F_{n-2} \text{ for } n \geqq 2.$$

We describe a way of deciding when a set S of m distinct numbers drawn from the sequence $F_2, F_3, F_4, \ldots$ corresponds to the sides of some plane polygon with m sides. If they do we call S a (scalene Fibonacci) polygon, for short.

To prove the result, we find it convenient to use the following identities, easily proved from (1) by induction on k and n, respectively:

(2) $$F_n = F_{n-2k} + \sum_{i=1}^{k} F_{n-2i+1} \quad \text{for } n \geqq 4 \text{ and } 0 < 2k < n,$$

(3) $$F_n > \sum_{i=2}^{n-2} F_i \quad \text{for } n \geqq 1 \text{ (the sum is zero if } n = 1, 2, 3).$$

Suppose once and for all that F_n is the largest number in S. If we denote by S(n, k) the set of numbers appearing in (2), then S is a polygon if and only if it properly contains some S(n, k). If it equals S(n, k) for some k we call it a degenerate polygon.

Proof: If $F_{n-1} \notin S$, then by (2) S contains no S(n, k). By (3) F_n exceeds the sum of the other numbers in S, which shows that S is not even a degenerate polygon. Now suppose that $F_{n-1} \in S$ (so that $n \geqq 3$) and proceed downward through the sequence in (1), starting with F_{n-1} and stopping short of F_1. The numbers alternate in and out of S until one of two things happens.

1. S is found to contain no S(n, k), either because the alternation stops at an adjacent pair not in S, say F_{n-2j}, F_{n-2j-1} with $n-2j-1 \geqq 2$, or continues to the bottom (here we set $n-2j-1 = 1$ or 0 according as n is even or odd). Then every number in S other than F_n occurs in either

$$\sum_{i=2}^{n-2j-2} F_i \quad \text{or} \quad \sum_{i=1}^{j} F_{n-2i+1} .$$

The first sum $< F_{n-2j}$ by (3), whence the sum of each $< F_n$ by (2). Thus S is again not even a degernate polygon.

2. The alternation stops with an adjacent pair in S, say F_{n-2k+1}, F_{n-2k} with $n-2k \geqq 2$, so that $S(n, k)$ is in S. Then (2) shows that S is a (degenerate) polygon if there are (no) numbers in S besides those in $S(n, k)$, on the grounds that F_n $(\leqq)$ the sum of the other numbers in S.

Could two sets of numbers drawn from F_2, F_3, F_4, ... be proportional to the lengths of the sides of a single polygon? This is not possible, at any rate, when the numbers in each set are distinct, for suppose that we did have two scalene Fibonacci polygons with largest sides F_n and F_N $(N > n)$ proportional to a third polygon, hence to each other, say in the ratio $P > 1$. We have just seen that if F_n is the largest number in such a set, then F_{n-1} must belong to it. Since the largest and second largest sides correspond, we have $PF_n = F_N$ and $PF_{n-1} = F_{N-1}$. By (1), we have, then, $PF_{n-2} = F_{N-2}$, and $n-2$ further applications of (1) yield finally $PF_0 = F_{N-n}$. By (1), the l. h. s. is zero and the r. h. s. positive, which is absurd.

An interesting exercise is to use this argument (with suitable amplification of the last sentence) on any two Fibonacci polygons such that in at least one of them there are numbers whose subscripts differ by only one or two. We need something stronger for such polygons as F_n, F_{n-3}, F_{n-3}, F_{n-3}, F_{n-3}, F_{n-3}.

The generalization of (2) which seems to be called for is some characterization of the coefficients in inequalities of the form

$$F_n \leqq \sum_{i=2}^{n} a_i F_i$$

where the a_i's are nonnegative integers.

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