LADDER NETWORK ANALYSIS USING POLYNOMIALS

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In this paper we develop some ideas with the recurring series

(1) $$B_n = k_1 B_{n-1} + k_2 B_{n-2}, \quad B_0 = 1, \ (k_1 \text{ and } k_2 \neq 0) \quad ,$$

and show a relationship between this sequence and the simple network of resistors known as a ladder-network.

The ladder-network in Figure 1 is an important network in communication systems. The m-L sections in cascade that make up this network can be characterized by defining:

(2) a) the attenuation (input voltage/output voltage) = A,

b) the output impedance = z_0,

c) the input impedance = z_1.

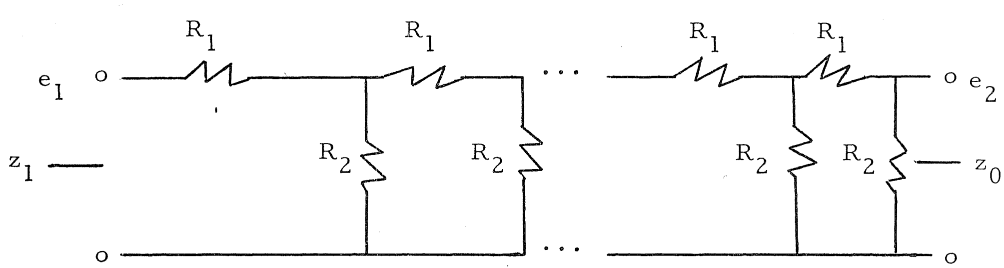

Figure 1

A result obtained by applying Kirchhoff's and Ohm's Laws to ladder-networks with m = 1, 2, 3, ..., $R_1 = R_2 k_1$, was tabulated with the results in Table 1, where setting $k_1 = 1$, $R_2 = 1$ ohm, the network in Figure 1 was analyzed by inspection [1] .

m	z_0	A	z_1
1	R_2	(k_1+1)	$(k_1+1)R_2$
2	$\left(\frac{k_1+1}{k+2}\right)R_2$	$(k_1^2+3k_1+1)$	$\left(\frac{k_1^2+3k_1+1}{k_1+2}\right)R_2$
3	$\left(\frac{k_1^2+3k_1+1}{k_1^2+4k_1+3}\right)R_2$	$(k_1^3+5k_1^2+6k_1+1)$	$\left(\frac{k_1^3+5k_1^2+6k_1+1}{k_1^2+4k_1+3}\right)R_2$
⋮	⋮	⋮	⋮

Table 1

We observe that the nth row in Table 1, may be written

m	z_0	A	z_1
n	$(C_{2n-2}/y_{2n-1})R_2$	C_{2n}	$(C_{2n}/y_{2n-1})R_2$

where,

(3) a) $C_n = k_1^{1/2}C_{n-1} + C_{n-2},\ C_0 = 1$,

b) $y_n = k_1^{1/2}y_{n-1} + y_{n-2},\ y_0 = 1/k_1^{1/2}$.

It then remains to solve for y_n and C_n in (3), to be able to analyze (Figure 1) by inspection for any value of k_1 ($k_1 \neq 0$), where $R_2 = 1$ ohm. So that, in (1), we let

(4) a) $w = (k_1 + (k_1^2 + 4k_2)^{1/2})/2$,

b) $v = (k_1 - (k^2 + 4k_2)^{1/2})/2$,

where it is evident,

c) $k_1 = w + v$,

and

d) $k_2 = -wv$.

Then, combining (c) and (d) with (1), leads to

(5)
$$B_n = ((w^2 - v^2)B_{n-1} - wv(w-v)B_{n-2})/(w-v) ,$$
$$B_n = ((w^3-v^3)B_{n-2} - wv(w^2-v^2)B_{n-3})/(w-v) ,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot ,$$
$$B_n = ((w^n - v^n)(w+v) - wv(w^{n-1} - v^{n-1})B_0)/(w-v) ,$$

and we have

(6) $$B_n = \frac{w^{n+1} - v^{n+1}}{w - v} .$$

Where, in (1) we replace k_1 with $k_1^{1/2}$ and k_2 with 1, and combining this result with (3) and (6), leads to

(7) a) $$C_n = \frac{(k_1^{1/2} + (k_1+4)^{1/2})^{n+1} - (k_1^{1/2} - (k_1+4)^{1/2})^{n+1}}{((k_1+4)^{1/2})2^{n+1}} = \phi(k_1),$$

and

b) $$y_n = \phi(k_1)/k_1^{1/2} .$$

(8) Theorem.

The attenuation (input voltage/output voltage = A) of m-L sections in cascade in a ladder-network is given by

$$A^2 = \sum_{r=0}^{2m-2} C_r((-C_{2m-1})/C_{2m-2})^r) .$$

The proof of the theorem rests on the following

(9) Lemma.

The power series

$$(-1)^n \sum_{r=0}^{n} B_r x^r ,$$

is always a square, where B_r is defined in (1).

Proof of lemma.

Let

(10) $$1 = (1 - k_1 x - k_2 x^2)(\sum_{r=0}^{n} B_r x^r) ,$$

then, by comparing coefficients and by (1), we have

(11) $$x = \frac{-(B_n k_1 + B_{n-1} k_2)}{B_n k_2} = \frac{-B_{n+1}}{B_n k_2} ,$$

and replacing x with $(-B_{n+1})/(B_n k_2)$ in $(1-k_1x-k_2x^2)$, leads to

(12) $$1-k_1x-k_2x^2 = (B_n^2 k_2 + B_n B_{n+1} k_1 - B_{n+1}^2)/(B_n^2 k_2) .$$

By (4, d) and (6) it is easily verified

(13) $$B_n^2 - B_{n+1}B_{n-1} = (-k_2)^n ,$$

so that

(14) $$B_n^2 k_2 + B_n B_{n+1} k_1 - B_{n+1}^2 = (-1)^n k_2^{n+1} .$$

Then, replacing the numerator in (12) by the result in (14) leads to

(15) $$1 - k_1 x - k_2 x^2 = ((-1)^n k_2^n)/B_n^2 ,$$

so that (10) may be written as

(16) $$(-1)^n B_n^2 = \sum_{r=0}^{n} B_r x^r ,$$

which completes the proof of the lemma.

(17) The proof of the theorem is immediate, when in (11) and (16), we replace n with 2m-2, k_1 with $k_1^{1/2}$, k_2 with 1, and combine the result with (7, a) and the values of the attenuation in Table 1.

REFERENCES

1. a) S. L. Basin, "The Appearance of Fibonacci Numbers and the Q Matrix in Electrical Network Theory," Math Mag., 36(1963) pp. 84-97.

b) S. L. Basin, "The Fibonacci Sequence as it Appears in Nature," Fibonacci Quarterly, 1(1963) pp. 54-55.

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