THE CHARACTERISTIC POLYNOMIAL OF A CERTAIN MATRIX OF BINOMIAL COEFFICIENTS

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1. Put

$$(1.1) \qquad A_{n+1} = \left[\binom{r}{n-s}\right] \quad (r,\ s = 0,\ 1,\ \ldots,\ n) \quad ,$$

a matrix of order $n+1$; for example

$$A_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 3 & 3 & 1 \end{bmatrix} .$$

Let

$$(1.2) \qquad f_{n+1}(x) = \det(xI - A_{n+1})$$

denote the characteristic polynomial of A_{n+1}. Hoggatt has communicated the following result to the writer.

Let $$F_o = 0,\ F_1 = 1,$$

$$F_{n+1} = F_n + F_{n-1} \qquad (n \geq 1)$$

denote the Fibonacci numbers. Define

$$(1.3) \qquad F_{n,r} = \frac{F_n F_{n-1} \cdots F_{n-r+1}}{F_1 F_2 \cdots F_r} \qquad (r \geq 1), \quad F_{n,o} = 1.$$

Then we have

$$(1.4) \qquad f_{n+1}(x) = \sum_{r=o}^{n+1} (-1)^{r(r+1)/2} F_{n+1,r}\, x^{n+1-r} \quad .$$

In the present paper we prove the truth of (1.4). Moreover we show that

$$(1.5) \qquad f_{n+1}(x) = \prod_{j=o}^{n} (x - \alpha^j \beta^{n-j}) \quad ,$$

where

(1.6) $$\alpha = (1+\sqrt{5})/2\,, \quad \beta = (1-\sqrt{5})/2 \quad .$$

Thus the characteristic values of A_{n+1} are

(1.7) $$\alpha^n, \ \alpha^{n-1}\beta, \ \ldots, \ \alpha\beta^{n-1}, \ \beta^n \quad .$$

Since they are distinct it follows that A_{n+1} is similar to a diagonal matrix.

2. We recall first that for any matrix A of the nth order with characteristic roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ we have

(2.1) $$\mathrm{tr}(A^k) = \lambda_1^k + \ldots + \lambda_n^k \quad (k = 0, 1, \ldots) \quad ,$$

where $\mathrm{tr}(A^k)$ denotes the trace of A^k. Moreover once these traces are known it is a simple matter to get the characteristic polynomial. We shall accordingly attempt to evaluate

(2.2) $$\mathrm{tr}(A_{n+1}^k) \quad (k = 0, 1, \ldots) \quad .$$

For $k = 1$ it is evident from (1.1) that

(2.3) $$\mathrm{tr}(A_{n+1}) = \sum_r \binom{r}{n-r} = F_{n+1} \quad .$$

For $k = 2$ we have

$$\mathrm{tr}(A_{n+1}^2) = \sum_{r,s} \binom{r}{n-s}\binom{s}{n-r} = \sum_{r,s} \binom{n-r}{s}\binom{n-s}{r}$$

$$= \sum_{r,s} \frac{(n-r)!\,(n-s)!}{r!\,s!\,((n-r-s)!)^2} = \sum_k \frac{n!}{k!\,(n-k)!} \sum_r \frac{(-k)_r(n-k+1)_r}{r!\,(-n)_r},$$

where

$$(a)_r = a(a+1)\ldots(a+r-1) \quad .$$

Since [1] page 37

$$\sum_r \frac{(-n)_r(a)_r}{r!\,(c)_r} = \frac{(c-a)_n}{(c)_n} \,,$$

we get

$$\operatorname{tr}(A_{n+1}^2) = \sum_k \frac{n!}{k!\,(n-k)!} \frac{(-2n+k-1)_k}{(-n)_k}$$

$$= \sum_k \frac{n!}{k!\,(n-k)!} \frac{(n-k)!}{n!} \frac{(2n-k+1)!}{(2n-2k+1)}$$

$$= \sum_k \binom{2n-k+1}{k} ,$$

so that

$$\operatorname{tr}(A_{n+1}^2) = F_{2n+2} . \tag{2.4}$$

In the next place we have

$$\operatorname{tr}(A_{n+1}^3) = \sum_{r,s,t} \binom{r}{n-s}\binom{s}{n-t}\binom{t}{n-r} = \sum_{r,s,t} \binom{n-r}{s}\binom{n-s}{t}\binom{n-t}{r} ,$$

but it does not seem possible to evaluate this sum by the above method. We shall instead employ the method used in [2] .

Starting with the identity

$$x^r(1+x)^{n-r} = \sum_s \binom{n-r}{s} x^{n-s} \tag{2.5}$$

replace x by $1+x^{-1}$. We get

$$(1+x)^r(1+2x)^{n-r} = \sum_{s,t} \binom{n-r}{s}\binom{n-s}{t} x^{n-t} . \tag{2.6}$$

Next multiply both sides by x^r and sum over r. This gives

$$\sum_{r=o}^{n} x^r(1+x)^r(1+2x)^{n-r} = \sum_{r,s,t} \binom{n-r}{s}\binom{n-s}{t} x^{n+r-t} .$$

The coefficient of x^n on the right is equal to

$$\sum_{r,s} \binom{n-r}{s}\binom{n-s}{r} = \sum_{r,s} \binom{r}{n-s}\binom{s}{n-r} = \operatorname{tr}(A_{n+1}^2) .$$

On the left we get

$$\sum_{r+s+t=n} \binom{r}{s}\binom{n-r}{s} 2^t = \sum_{r+s\leq n} \binom{r}{s}\binom{n-r}{s} 2^{n-r-s} = u_n ,$$

say. Then

$$\sum_{n=o}^{\infty} u_n x^n = \sum_{r,\, s=o}^{\infty} \binom{r}{s} x^{r+s} \sum_{n=r+s}^{\infty} \binom{n-r}{s} (2x)^{n-r-s}$$

$$= \sum_{r,\, s=o}^{\infty} \binom{r}{s} x^{r+s} (1-2x)^{-s-1}$$

$$= \sum_{s=o}^{\infty} x^{2s} (1-x)^{-s-1} (1-2x)^{-s-1}$$

$$= \frac{1}{1-3x+x^2} = \frac{1}{\alpha^2 - \beta^2}\left(\frac{\alpha^2}{1-\alpha^2 x} - \frac{\beta^2}{1-\beta^2 x}\right)$$

where α, β are defined by (1.6). We have therefore

$$u_n = \frac{\alpha^{2n+2} - \beta^{2n+2}}{\alpha^2 - \beta^2} = \frac{F_{2n+2}}{F_2} = F_{2n+2}$$

in agreement with (2.4).

Returning to (2.6), again replace x by $1+x^{-1}$. We find that

$$(2.7) \qquad (1+2x)^r (2+3x)^{n-r} = \sum_{s,\, t,\, j} \binom{n-r}{s}\binom{n-s}{t}\binom{n-t}{j} x^{n-j} .$$

Multiply by x^r and sum over r. We get

$$(2.8) \qquad \sum_{r=o}^{n} x^r (1+2x)^r (2+3x)^{n-r} = \sum_{r,\, s,\, t,\, j} \binom{n-r}{s}\binom{n-s}{t}\binom{n-t}{j} x^{n+r-j} .$$

The coefficient of x^n on the right of (2.8) is evidently

$$\sum_{r,\, s,\, t} \binom{n-r}{s}\binom{n-s}{t}\binom{n-t}{r} = \mathrm{tr}(A_{n+1}^3) .$$

On the left we get

$$\sum_{r+s+t=n} \binom{r}{s}\binom{n-r}{t} 2^s 2^{n-r-t} 3^t = \sum_{r+s\leq n} \binom{r}{s}\binom{n-r}{s} 2^{2s} 3^{n-r-s} = u_n ,$$

say. Then as above

$$\sum_{n=o}^{\infty} u_n x^n = \sum_{r,\, s=o}^{\infty} \binom{r}{s} 2^{2s} x^{r+s} (1-3x)^{-s-1}$$

$$= \sum_{s=o}^{\infty} (2x)^{2s} (1-x)^{-s-1} (1-3x)^{-s-1} = \frac{1}{(1-x)(1-3x)-4x^2}$$

$$= \frac{1}{1-4x-x^2} = \frac{1}{(1-\alpha^3 x)(1-\beta^3 x)}$$

$$= \frac{1}{\alpha^3-\beta^3}\left(\frac{\alpha^3}{1-\alpha^3 x} - \frac{\beta^3}{1-\beta^3 x}\right),$$

so that

$$u_n = \frac{\alpha^{3n+3} - \beta^{3n+3}}{\alpha^3 - \beta^3} = \frac{F_{3n+3}}{F_3} .$$

It follows that

$$\text{(2.9)} \qquad \operatorname{tr}(A_{n+1}^3) = \frac{F_{3n+3}}{F_3} .$$

3. We are now able to handle the general case. In (2.6) replace x by $1+x^{-1}$ and we get

$$\text{(3.1)} \qquad (2+3x)^r (3+5x)^{n-r} = \sum_{s,\, t,\, j,\, k} \binom{n-r}{s}\binom{n-s}{t}\binom{n-t}{j}\binom{n-j}{k} x^{n-k} .$$

The general formula of this type is

$$\text{(3.2)} \qquad (F_{k-1}+xF_k)^r (F_k+xF_{k+1})^{n-r}$$

$$= \sum_{r_1, \ldots, r_k} \binom{n-r}{r_1}\binom{n-r_1}{r_2}\cdots\binom{n-r_{k-1}}{r_k} x^{n-r_k} \quad (k=1, 2, 3, \ldots) .$$

Indeed for $k = 1, 2, 3, 4$, (3.2) reduces to (2.5), (2.6), (2.7), (3.1), respectively. Assuming that (3.2) holds for the value k we replace x by $1+x^{-1}$ and multiply the result by x^n. The left member becomes

$$(xF_{k-1} + xF_k + F_k)^r(xF_k + xF_{k+1} + F_{k+1})^{n-r}$$

$$= (F_k + xF_{k+1})^r(F_{k+1} + xF_{k+2})^{n-r} ,$$

while the right member becomes

$$\sum_{r_1,\ldots,r_{k+1}} \binom{n-r}{r_1}\binom{n-r_1}{r_2}\cdots\binom{n-r_{k-1}}{r_k}\binom{n-r_k}{r_{k+1}} x^{n-r_{k+1}} .$$

This evidently completes the proof of (3.2).

Next multiply (3.2) by x^r and sum over r. This gives

$$(3.3) \qquad \sum_{r=0}^{n} x^r(F_{k-1}+xF_k)^r(F_k+xF_{k+1})^{n-r}$$

$$= \sum_{r,r_1,\ldots,r_k} \binom{n-r}{r_1}\binom{n-r_1}{r_2}\cdots\binom{n-r_{k-1}}{r_k} x^{n+r-r_k} .$$

The coefficient of x^n on the right of (3.3) is equal to

$$\sum_{r+s+t=n} \binom{r}{s}\binom{n-r}{t} F_{k-1}^{r-s} F_k^s F_k^{n-r-t} F_{k+1}^t$$

$$= \sum_{r+s\le n} \binom{r}{s}\binom{n-r}{s} F_{k-1}^{r-s} F_k^{2s} F_{k-1}^{n-r-s} = u_n^{(k)} ,$$

say. Then

$$\sum_{n=0}^{\infty} u_n^{(k)}x^n = \sum_{r,s=0}^{\infty} \binom{r}{s}F_{k-1}^{r-s} F_k^{2s} x^{r+s}(1-F_{k+1}x)^{-s-1}$$

$$= \sum_{s=0}^{\infty} F_k^{2s} x^{2s}(1-F_{k-1}x)^{-s-1}(1-F_{k+1}x)^{-s-1}$$

$$= \frac{1}{(1-F_{k-1}x)(1-F_{k+1}x)-F_k^2x^2} = \frac{1}{1-(F_{k-1}+F_{k+1})x+(-1)^kx^2} .$$

But

$$1-(F_{k-1}+F_{k+1})x+(-1)^k x^2 = 1-(\alpha^k+\beta^k)x+(\alpha\beta)^k x^2 = (1-\alpha^k x)(1-\beta^k x)$$

and

$$\frac{1}{(1-\alpha^k x)(1-\beta^k x)} = \frac{1}{\alpha^k-\beta^k}\left(\frac{\alpha^k}{1-\alpha^k x} - \frac{\beta^k}{1-\beta^k x}\right).$$

It follows that

$$u_n^{(k)} = \frac{\alpha^{nk+k} - \beta^{nk+k}}{\alpha^k - \beta^k} = \frac{F_{nk+k}}{F_k} .$$

Comparison with (3.4) yields

$$\text{(3.5)} \qquad \operatorname{tr}(A_{n+1}^k) = \frac{F_{nk+k}}{F_k} .$$

4. We now return to the characteristic polynomial

$$f_{n+1}(x) = \det(xI - A_{n+1}) .$$

If we denote the characteristic values by $\lambda_o, \lambda_1, \ldots, \lambda_n$, we have

$$\frac{f'_{n+1}(x)}{f_{n+1}(x)} = \sum_{j=o}^{n} \frac{1}{x-\lambda_j} = \sum_{k=o}^{\infty} x^{-k-1} \sum_{j=o}^{n} \lambda_j^k = \sum_{k=o}^{\infty} x^{-k-1} \operatorname{tr}(A_{n+1}^k)$$

$$= \sum_{k=o}^{\infty} x^{-k-1} \frac{\alpha^{nk+k}-\beta^{nk+k}}{\alpha^k - \beta^k} = \sum_{k=o}^{\infty} x^{-k-1} \sum_{j=o}^{n} \alpha^{jk}\beta^{(n-j)k}$$

$$= \sum_{j=o}^{n} \frac{1}{x-\alpha^j\beta^{n-j}} .$$

It follows that

$$\text{(4.1)} \qquad f_{n+1}(x) = \prod_{j=o}^{n} (x-\alpha^j\beta^{n-j})$$

and therefore the characteristic values of A_{n-1} are the numbers

$$\text{(4.2)} \qquad \alpha^n, \alpha^{n-1}\beta, \ldots, \alpha\beta^{n-1}, \beta^n .$$

We shall now show that

$$(4.3)\qquad \prod_{j=0}^{n} (x-\alpha^j\beta^{n-j}) = \sum_{r=0}^{n+1} (-1)^{r(r+1)/2} F_{n+1,r}\, x^{n+1-r} ,$$

with $F_{n+1,r}$ defined by (1.3).

To prove (4.3) we make use of the familiar identity

$$(4.4)\qquad \prod_{j=0}^{n-1} (1-q^j x) = \sum_{r=0}^{n} (-1)^r q^{r(r-1)/2} \left[{n \atop r}\right] x^r ,$$

where

$$(4.5)\qquad \left[{n \atop r}\right] = \frac{(1-q^n)(1-q^{n-1})\dots(1-q^{n-r+1})}{(1-q)(1-q^2)\dots(1-q^r)} .$$

If we replace q by β/α we find that

$$\left[{n \atop r}\right] \rightarrow \alpha^{r^2-nr} F_{nr} .$$

Thus (4.4) becomes

$$\prod_{j=0}^{n-1} (1-\alpha^{-j}\beta^j x) = \sum_{r=0}^{n} (-1)^r \alpha^{r(r+1)/2\,-nr} \beta^{r(r-1)/2} F_{n,r}\, x^r .$$

Now replace x by $\alpha^{n-1}x$. Then

$$\prod_{j=0}^{n-1} (1-\alpha^{n-j-1}\beta^j x) = \sum_{r=0}^{n} (-1)^r (\alpha\beta)^{r(r-1)/2} F_{n,r}\, x^r$$

$$= \sum_{r=0}^{n} (-1)^{r(r+1)/2} F_{n,r}\, x^r .$$

Replacing x by x^{-1} we get

$$\prod_{j=0}^{n-1} (x-\alpha^{n-j-1}\beta^j) = \sum_{r=0}^{n} (-1)^{r(r+1)/2} F_{n,r}\, x^{n-r} .$$

This evidently proves (4.3).

Incidentally we have proved the stronger result that (4.3) holds when α, β are any numbers such that $\alpha\beta = -1$ and

$$F_{n,r} = \frac{(\alpha^n-\beta^n)(\alpha^{n-1}-\beta^{n-1})\ldots(\alpha^{n-r+1}-\beta^{n-r+1})}{(\alpha-\beta)(\alpha^2-\beta^2)\ldots(\alpha^r-\beta^r)} .$$

If we now compare (4.3) with (4.1) it is clear that we have proved (1.4).

5. It is of interest to note that the particular characteristic values α^n, β^n can be predicted directly as follows. We have

$$\sum_s \binom{r}{n-s} \alpha^s = \alpha^n \sum_s \binom{r}{n-s} \alpha^{s-n}$$

$$= \alpha^n(1+\alpha^{-1})^r = \alpha^{n-r}(\alpha+1)^r = \alpha^{n+r} .$$

This shows that $[1, \alpha, \ldots, \alpha^n]$ is the characteristic vector corresponding to α^n. Similarly $[1, \beta, \ldots, \beta^n]$ is the characteristic vector corresponding to β^n.

However it is not evident how to find the remaining characteristic vectors when $n > 1$. We can for example show that there are no other characteristic vectors of the type $[1, \gamma, \ldots, \gamma^n]$. Indeed assume that

(5.1) $$\sum_s \binom{r}{n-s} \gamma^s = \lambda\gamma^r \quad (r = 0, 1, \ldots, n) .$$

Then since

$$\sum_s \binom{r}{n-s} \gamma^s = \gamma^n(1+\gamma^{-1})^r = \gamma^{n-r}(\gamma+1)^r ,$$

it follows from (5.1) that

$$\gamma^{-2r}(\gamma+1)^r = \lambda\gamma^{-n} \quad (r = 0, 1, \ldots, n) .$$

Since the right side is independent of r we must have $\gamma + 1 = \gamma^2$, so that $\gamma = \alpha$ or β.

REFERENCES

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2. L. Carlitz, "A Binomial Identity Arising from a Sorting Problem," Siam Review, Vol. 6(1964), pp. 20-30.

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