THE CHARACTERISTIC POLYNOMIAL OF THE GENERALIZED SHIFT MATRIX

V.E. HOGGATT, JR. and A.P. HILLMAN
San Jose State College and University of Santa Clara

T. A. Brennan [3] obtained the characteristic polynomial for the k by k matrix $P_k = [P_{ij}]$ with the binomial coefficient $\binom{i-1}{k-j}$ as the element P_{ij} in the i-th row and j-th column. See [6] and [7] for special cases. L. Carlitz [5] used another method involving some very interesting identities to achieve the same result. In this paper we find the characteristic polynomial for a generalization of the P_k.

Let F be a field of characteristic zero, let p and q be in F, and let

$$y_{n+2} = qy_n + py_{n+1}, \quad q \neq 0 \tag{1}$$

be a second order homogeneous linear difference equation over F. We restrict n to be an integer in (1). Let a and b be the zeros of the auxiliary polynomial

$$x^2 - px - q = (x - a)(x - b)$$

of (1). We deal only with the case in which (1) is <u>ordinary</u> in the sense of R. F. Torretto and J. A. Fuchs [4], i.e., we assume that either $a = b$ or $a^n \neq b^n$ for all positive integers n. Using the notation of E. Lucas [1] we let U_n be the solution $(a^n - b^n)/(a - b)$ of (1). Also we use the notation of [3] and [4] for the generalized binomial coefficient

$$\begin{bmatrix} m \\ j \end{bmatrix} = \frac{U_m U_{m-1} \cdots U_{m-j+1}}{U_1 U_2 \cdots U_j}, \quad \begin{bmatrix} m \\ 0 \end{bmatrix} = 1$$

of D. Jarden [2].

Jarden showed that the product z_n of the n-th terms of k-1 solutions of (1) satisfies

$$\sum_{h=0}^{k} (-1)^h \begin{bmatrix} k \\ h \end{bmatrix} (-q)^{h(h-1)/2} z_{n-h} = 0 . \tag{2}$$

Torretto and Fuchs showed that (1) is ordinary if and only if the "sequences" (i.e., functions of the integral variable n)

$$z_n(i,k) = U_n^{k-i}U_{n+1}^{i-1}\ ; \quad i = 1, 2, \ldots, k \tag{3}$$

form a basis for the vector space of all sequences satisfying (2).

Let $C_n = C_n(k)$ be the k-dimensional column vector with $z_n(i,k)$ the element in the i-th row and let $S = S(k)$ be the k by k matrix $\left[s_{ij}\right]$ with

$$s_{ij} = \binom{i-1}{k-1} q^{k-j}p^{i+j-k-1} \quad . \tag{4}$$

We show below that S has the shifting property $SC_n = C_{n+1}$ and that the characteristic polynomial of S is the auxiliary polynomial

$$f(X) = \sum_{h=0}^{k} (-1)^h \left[{k \atop h}\right](-q)^{h(h-1)/2}X^{n-h} \tag{5}$$

of the difference equation (2).

Using (3) and (1) we have,

$$z_{n+1}(i,k) = U_{n+1}^{k-i}(qU_n + pU_{n+1})^{i-1} = \sum_{h=0}^{i-1} \binom{i-1}{h} q^h p^{i-1-h} U_n^h U_{n+1}^{k-1-h} \quad . \tag{6}$$

Letting $h = k - j$ in (6) and reversing the order of the terms leads to

$$z_{n+1}(i,k) = \sum_{j=k+1-i}^{k} \binom{i-1}{k-j} q^{k-j}\, p^{i+j-k-1}\, U_n^{k-j}\, U_{n+1}^{j-1} \quad . \tag{7}$$

Using (4) and the fact that $\binom{m}{r} = 0$ for $m < r$, we can rewrite (7) as

$$z_{n+1}(i,k) = \sum_{j=1}^{k} s_{ij}\, z_n(j,k) \quad . \tag{8}$$

Let $T = \left[t_{ij}\right]$ be the matrix $f(S)$, where $f(X)$ is as defined in (5). In matrix notation (8) is $SC_n = C_{n+1}$. By induction it follows that $S^iC_n = C_{n+i}$. Since the elements of the C_n in a fixed position satisfy the difference equation (2), so do the vectors C_n. This is equivalent to $TC_n = 0$ for all integers n, i.e.,

(9) $$t_{i1}\, z_n(1,k) + t_{i2}\, z_n(2,k) + \ldots + t_{ik}\, z_n(k,k) = 0$$

for all n. Since it was proved in [3] that the sequences

$$z_n(1,k), \ \ldots, \ z_n(k,k)$$

are linearly independent, (9) implies that each $t_{ij} = 0$. Hence $T \equiv 0$ and we have shown that S satisfies $f(X) = 0$. Let $g(X) = 0$ be the monic polynomial equation of least degree over K for which $g(S) = 0$. Then $g(X)$ divides $f(X)$.

Clearly the last column of S is C_1. Since only the last column of S^n is involved in finding the last column of S^{n+1} by the formula $S \cdot S^n = S^{n+1}$ and since $SC_n = C_{n+1}$, it follows by induction that the last column of S^n is C_n. In particular, the element in the first row and k-th column of S^n is $z_n(1,k)$, which we shorten to z_n in what follows. By definition

$$z_n = U_n^{k-1} = \left[(a^n - b^n)/(a - b)\right]^{k-1} .$$

Expanding the binomial $(a^n - b^n)^{k-1}$ we see that

(10) $$z_n = c_1(a^{k-1})^n + c_2(a^{k-2}b)^n + \ldots + c_k(b^{k-1})^n$$

with each c_h different from zero.

Since $g(S) = 0$, the elements in the S^n in a fixed position, and in particular the z_n, satisfies the difference equation for which $g(x)$ is the auxiliary polynomial. Jarden showed in [5] that the zeros of $f(x)$ are

(11) $$a^{k-1},\ a^{k-2}b,\ a^{k-3}b^2,\ \ldots,\ b^{k-1} .$$

The zeros of $g(x)$ thus are some or all of these zeros of $f(x)$. If $f(x) \neq g(x)$, then $g(x)$ has lower degree than $f(x)$ and so

$$z_n = d_1 r_1^n + d_2 r_2^n + \ldots + d_m r_m^n$$

with $m < k$, the d_i in F, and each r_i one of the elements of (11). Since no c_h in (10) is zero, this would mean that (10) is not unique and hence that the sequences $(a^h b^{k-1-h})^n$, $0 \leq h \leq k-1$, are linearly dependent. As in [4], this would contradict the fact that (1) is ordinary. Hence $f(X) \equiv g(X)$. Since the characteristic polynomial $\phi(X)$ of S is monic, of degree k, and a multiple of $g(X)$, $\phi(X)$ must also be $f(X)$ and (11) gives the characteristic values of S. This completes the proof.

REFERENCES

1. E. Lucas, Théorie des Fonctions Numériques Simplement Périodic, Amer. Jour. of Math., 1(1878) 184-240 and 289-321.
2. D. Jarden, "Recurring Sequences," published by Riveon Lematimatika, Jerusalem (Israel), 1958.
3. T. A. Brennan, "Fibonacci Powers and Pascal's Triangle in a Matrix," Fibonacci Quarterly, 2(1964) pp. 93-103 and 177-184.
4. R. F. Torretto and J. A. Fuchs, "Generalized Binomial Coefficients," Fibonacci Quarterly, 2(1964) pp. 296-302.
5. L. Carlitz, "The Characteristic Polynomial of a Certain Matrix of Binomial Coefficients, Fibonacci Quarterly, 3(1965) pp.
6. V. E. Hoggatt, Jr. and Marjorie Bicknell, "Fourth Power Fibonacci Identities From Pascal's Triangle," Fibonacci Quarterly 2(1964) Dec., pp. 261-266.
7. V. E. Hoggatt, Jr. and Marjorie Bicknell, "Some New Fibonacci Identities," Fibonacci Quarterly 2(1964) February, pp. 29-32.

xxxxxxxxxxxxxxx

(Continued from page 134)

A more extensive analysis of the generated compositions which yield Fibonacci numbers will be jointly attempted by Dr. Hoggatt and the author in a subsequent paper. In addition, the author is planning to submit some papers in the future, which will furnish some original models and theorems connected with Fibonacci numbers and their properties. These models and theorems have been incorporated <u>in part</u> in the author's doctoral thesis, which has been cited as a reference in this article.