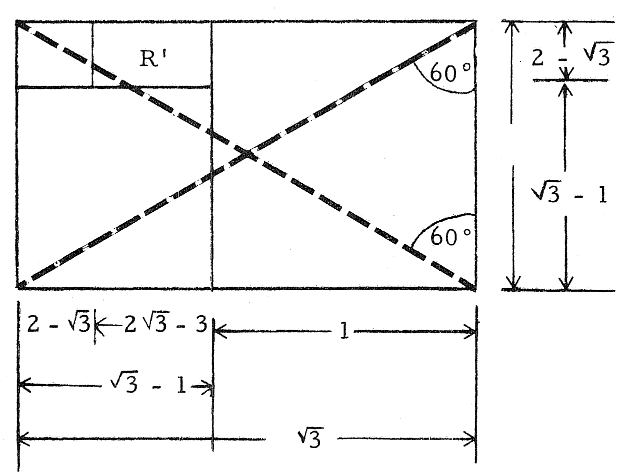
A NEAR-GOLDEN RECTANGLE AND RELATED RECURSIVE SERIES

MARJORIE BICKNELL AND JAMES LEISSNER
Adrian C. Wilcox High School, Santa Clara, California
Thomas Jefferson High School, San Antonio, Texas

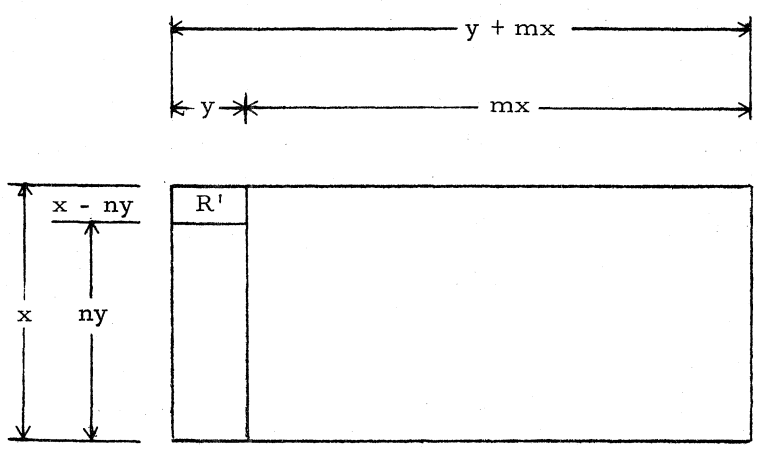
The rectangle whose diagonals form equilateral triangles with its widths has some surprising properties, including a related Fibonacci-like series of integers. Before discussing this rectangle, for later comparison, we call to mind another rectangle. The famous Golden Rectangle has the property that when a full-width square is cut from one end, the remaining part has the same proportions as the original rectangle, the ratio of length to width being $(1 + \sqrt{5})/2$. Joseph Raab discussed other golden-type rectangles [1], which have the property that when an integral number k of full-width squares are cut from one end, the remaining part has the same proportions as the original rectangle. These golden-type rectangles also have related series of integers.

In the rectangle whose diagonals form equilateral triangles with its widths, the ratio of length to width is $\sqrt{3}$, certainly not "golden." But after cutting a full-width square from one end, there appears a glitter as the ratio of length to width becomes $(1 + \sqrt{3})/2$. Operating similarly on this rectangle, the ratio becomes $\sqrt{3} + 1$, and repeating the process one last time makes the ratio of length to width again $\sqrt{3}$.



Some more "near-golden" rectangles appear as more general cases of removing squares of the width in a rectangle to obtain rectangles similar to the original. To simplify the discussion, we will designate a rectangle by a capital letter and its ratio of length to width by the corresponding small letter.

From a rectangle R with width x and length $y + mx$, remove the total number m of full-width squares contained in R to obtain rectangle P. From P, remove the total number n of full-width squares contained in P to form rectangle R'.



If R' is similar to R, then $r' = r$ so that $y/(x - ny) = (y + mx)/x$. Solving for x/y 1 p, we find

$$r' = r = (mn + \sqrt{m^2n^2 + 4mn})/2n,$$

$$p = (mn + \sqrt{m^2n^2 + 4mn})/2m,$$

(Note that $R:R' = rp$, and that $m = n = 1$ yields the Golden Rectangle.)

When we cut full-width squares from P, if we remove an integral number n less than the total number of full-width squares available, and if R' and R are similar,

$$r = (\sqrt{(m+n)^2 + 4} + m - n)/2,$$

$$p = (\sqrt{(m+n)^2 + 4} + m + n)/2.$$

(Note again the Golden Rectangle for $m = 1$ and $n = 0$, when $P = R'$.)

Suppose that we remove the full amount of available full-width squares in forming P and R', but R' and R are not similar. If a rectangle T, similar to R, can be obtained from R' by the removal of an integral number q of squares of the width of R', then

$$r = t = (\sqrt{n^2(m+q)^2 + 4n(m+q)} + n(m - q))/2n,$$

$$p = (\sqrt{n^2(m+q)^2 + 4n(m+q)} + n(m + q))/2(m + q),$$

$$r' = (\sqrt{n^2(m+q)^2 + 4n(m+q)} + n(m + q)/2n.$$

Again, $q = 0$ and $m = 1$ yields the Golden Rectangle, with $r = p = r' = (1 + \sqrt{5})/2$. Also, $q = m = n = 1$ yields (for R and T) the rectangle with diagonals forming equilateral triangles with its widths, with $p = (1 + \sqrt{3})/2$.

The similarity of form between the ratio $(1 + \sqrt{3})/2$, hereafter called θ, and the golden ratio given above, suggests that we seek a Fibonacci-type series associated with powers of θ. Consider the following:

$$\begin{aligned}
\theta &= (1 + \sqrt{3})/2 &&= (1)\theta + 0\\
\theta^2 &= (2 + \sqrt{3})/2 &&= (1)\theta + 1/2\\
\theta^3 &= (5 + 3\sqrt{3})/4 &&= (3/2)\theta + 1/2\\
\theta^4 &= (7 + 4\sqrt{3})/4 &&= (4/2)\theta + 3/4\\
\theta^5 &= (19+11\sqrt{3})/8 &&= (11/4)\theta + (4/4)\\
\theta^6 &= (26+15\sqrt{3})/8 &&= (15/4)\theta + (11/8).
\end{aligned}$$

The numerators of either the coefficients of θ or the constant addends and the coefficients of $\sqrt{3}$ form the following series: 1, 1, 3, 4, 11, 15, 41, 56, ... It can be proved by induction that this series is defined by

$$P_{2n} = P_{2n-1} + P_{2n-2}$$

$$P_{2n+1} = 2P_{2n} + P_{2n-1}, \quad n = 1, 2, \ldots,$$

where $P_1 = P_2 = 1$. A second series: 1, 2, 5, 7, 19, 26, ..., having the same recursion formulas as the above, appears in the computation of powers of θ. We shall call the nth term in the second series R_n.

If $\theta = (1 + \sqrt{3})/2$ and $\phi = (1 - \sqrt{3})/2$, it is not difficult to show by induction that

$$P_n = (\theta^n - \phi^n)/\sqrt{3}\cdot 2^{[1-n/2]},$$

$$R_n = (\theta^n + \phi^n)/2^{[1-n/2]}, \quad n = 1, 2, 3, \ldots,$$

where $[x]$ is the largest integer in x. The series just defined bear a striking resemblance to the Fibonacci and Lucas series as defined by the Binet formula in terms of the golden ratio, where the nth Fibonacci and nth Lucas number are given respectively by

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}, \quad L_n = \alpha^n + \beta^n \text{ for } \alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2}.$$

Use of the above form for P_n and R_n and standard limit theorems leads to

$$\lim_{n\to\infty} P_{2n}/P_{2n-1} = \theta \quad\text{and}\quad \lim_{n\to\infty} R_{2n}/R_{2n-1} = \phi;$$

$$\lim_{n\to\infty} P_{2n+1}/P_{2n} = 2\theta \quad\text{and}\quad \lim_{n\to\infty} R_{2n+1}/R_{2n} = 2\phi.$$

Finally, as n increases, R_n/P_n oscillates about its limit, $\sqrt{3}$.

Also established by induction are forms for powers of θ.

$$\theta^n = (P_n\theta)/2^{[(n-1)/2]} + P_{n-1}/2^{[n/2]} = (R_n + P_n\sqrt{3})/2^{[(n+1)/2]}$$

and

$$\theta^{-n} = (-2)^n\left(P_{n+1}/2^{[n/2]} - P_n\theta/2^{[(n-1)/2]}\right).$$

For comparison, if

$$\frac{1+\sqrt{5}}{2} = \alpha, \text{ then } \alpha^n = (L_n + F_n\sqrt{5})/2,$$

where F_n is the nth Fibonacci number and L_n the nth Lucas number.

Other theorems, also possible to establish by induction, are:

$$\sum_{i=1}^{2n} P_i = P_{2n+1} - (P_{2n-1} + 1)/2,$$

$$\sum_{i=1}^{2n+1} P_i = (P_{2n+3} - 1)/2,$$

$$\sum_{i=1}^{2(2n-1)} P_i = P_{2n}P_{2n+1} - P_{2n-1}^2$$

$$P_nP_{n+3} - P_{n+1}P_{n+2} = (-1)^{n+1}.$$

Considering the even ordered elements and the odd ordered elements of the series separately leads to

$$P_{2n} = 4P_{2n-2} - P_{2n-4}$$

$$P_{2n+1} = 4P_{2n-1} - P_{2n-3},$$

which in turn can be used to prove the following relationships between R_n and P_n, and summation formulas for even or odd elements of the series P_n:

$$R_{2n} = P_{2n-1} + P_{2n},$$

$$3P_{2n} = R_{2n-1} + R_{2n};$$

$$\sum_{i=1}^{n} P_{2i} = (P_{2n+1} - 1)/2 = (3P_{2n} - P_{2n-2} - 1)/2,$$

and

$$\sum_{i=1}^{n} P_{2i-1} = P_{2n} = (P_{2n+3} - P_{2n+1})/2.$$

REFERENCES

1. Joseph A. Raab, "A Generalization of the Connection Between the Fibonacci Sequence and Pascal's Triangle," Fibonacci Quarterly, 3:1, Oct., 1963, pp. 21-32.

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