ELEMENTARY PROBLEMS AND SOLUTIONS

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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets in the format used below. Solutions should be received within two months of publication.

B-70 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*

Denote x^a by ex(a). Show that the following expression, containing n integrals,

$$\int_0^1 \mathrm{ex}\,(\int_0^1 \mathrm{ex}(\int_0^1 \mathrm{ex}(\ldots\int_0^1 \mathrm{ex}(\int_0^1 x\,dx)dx)\ldots dx)dx)dx$$

equals F_{n+1} / F_{n+2}, where F_n is the n-th Fibonacci number.

B-71 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*

Find $a^{-2} + a^{-3} + a^{-4} + \ldots$, where $a = (1 + \quad 5)/2$.

B-72 *Proposed by J. A. H. Hunter, Toronto, Canada*

Each distinct letter in this simple alphametic stands for a particular and different digit. We all know how rabbits link up with the Fibonacci series, so now evaluate our RABBITS.

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RABBITS
   BEAR
RABBITS
     AS
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ASERIES
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B-73 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*

Prove that

$$\sum_{k=0}^{n} \sum_{j=0}^{n} \binom{n}{k} \binom{k+r-j-1}{j} = 1 + \sum_{m=0}^{2n+r-2} \sum_{p=0}^{m} \binom{m-p-1}{p} ,$$

where $\binom{n}{r} = 0$ for $n < r$.

B-74 *Proposed by M. N. S. Swamy, University of Saskatchewan, Regina, Canada*

The Fibonacci polynomial $f_n(x)$ is defined by $f_1 = 1$, $f_2 = x$, and $f_n(x) = xf_{n-1}(x) + f_{n-2}(x)$ for $n > 2$. Show the following:

(a) $$x \sum_{r=1}^{n} f_r(x) = f_{n+1} + f_n - 1 .$$

(b) $$f_{m+n+1} = f_{m+1} f_{n+1} + f_m f_n .$$

(c) $$f_n(x) = \sum_{j=0}^{[(n-1)/2]} \binom{n-j-1}{j} x^{n-2j-1} ,$$

where $[k]$ is the greatest integer not exceeding k. Hence show that the n-th Fibonacci number

$$F_n = \sum_{j=0}^{[(n-1)/2]} \binom{n-j-1}{j} .$$

B-75 *Proposed by M. N. S. Swamy, University of Saskatchewan, Regina, Canada*

Let $f_n(x)$ be as defined in B-74. Show that the derivative

$$f'_n(x) = \sum_{r=1}^{n-1} f_r(x) f_{n-r}(x) \text{ for } n > 1.$$

SOLUTIONS

ONE, TWO, THREE — OUT

B-58 *Proposed by Sidney Kravitz, Dover, New Jersey*

Show that no Fibonacci number other than 1, 2, or 3 is equal to a Lucas number.

Solution by Douglas Lind, University of Virginia, Charlottesville, Va.

Since $L_k = F_{k-1} + F_{k+1}$, the assertion is equivalent to

(1) $$F_n = F_{k-1} + F_{k+1}.$$

If $k \geq 3$, the $n > k+1$ and (1) is clearly impossible since

$$F_{k-1} + F_{k+1} < F_k + F_{k+1} = F_{k+2} \leq F_n.$$

Impossibility for $k \geq 3$ implies impossibility for $k \leq -3$ since only signs are different. For $-3 < k < 3$ we find $F_{-2} = L_{-1} = 1$, $F_3 = l_0 = 2$, $F_1 = L_1 = 1$, and $F_4 = L_2 = 3$, corresponding to $k = -1$, 0, 1, and 2 respectively. Hence these are the only solutions. (The crux of this problem is solved in the discussion of equation (12) in Carlitz' "A Note on Fibonacci Numbers," this Quarterly 1 (1964) No. 2 pp. 15-28).

Also solved by J. L. Brown, Jr.; Gary C. McDonald; C. B. A. Peck; and the proposer.

B-59 *Proposed by Brother U. Alfred, St. Mary's College, California*

Show that the volume of a truncated right circular cone of slant height F_n with F_{n-1} and F_{n+1} the diameters of the bases is

$$\sqrt{3}\,\pi(F_{n+1}^3 - F_{n-1}^3)/24.$$

Solution by Douglas Lind, University of Virginia, Charlottesville, Va.

It is well-known that if h is the height of the frustrum of a right circular cone, s the slant height, and r_1 and r_2 the radii of the bases, then the volume V is

$$V = (\pi h/3)(r_1^2 + r_1 r_2 + r_2^2)$$

$$= (\pi/3)\sqrt{s^2 - (r_2 - r_1)^2}\,(r_1^2 + r_1 r_2 + r_2^2).$$

For this problem $r_1 = F_{n-1}/2$, $r_2 = F_{n+1}/2$ and $s = F_n$, so that

$$V = \frac{\pi}{3}\sqrt{F_n^2 - (F_{n+1} - F_{n-1})^2/4}\,(F_{n-1}^2 + F_{n-1}F_{n+1} + F_{n+1}^2)/4$$

$$= \pi\sqrt{F_n^2 - F_n^2/4}\,(F_{n-1}^2 + F_{n-1}F_{n+1} + F_{n+1}^2)/12$$

$$= \sqrt{3}\,\pi F_n\,(F_{n-1}^2 + F_{n-1}F_{n+1} + F_{n+1}^2)/24$$

$$= \sqrt{3}\pi(F_{n+1} - F_{n-1})(F_{n-1}^2 + F_{n-1}F_{n+1} + F_{n+1}^2)/24$$

$$= \sqrt{3}\pi(F_{n+1}^3 - F_{n-1}^3)/24.$$

We remark that the area A of the curved surface of the frustrum is

$$A = \pi\, F_n(F_{n+1} + F_{n-1})/2 = (\pi/2)F_nL_n.$$

Also solved by Carole Bania, Gary C. McDonald, Kenneth E. Newcomer, C. B. A. Peck, M. N. S. Swamy, Howard L. Walton, John Wessner, Charles Ziegenfus, and the proposer. McDonald also added the formula for the curved surface.

B-60 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Show that $L_{2n}L_{2n+2} + 5F_{2n+1}^2 = 1$, where F_n and L_n are the n-th Fibonacci number and Lucas number, respectively.

Solution by 2nd Lt. Charles R. Wall, U. S. Army, A. P. O., San Francisco, Calif.

Using my second answer to B-22 (Vol. 2, No. 1, p. 78),

$$\begin{aligned} L_{2(n+1)}\, L_{2n} &= 5F_{(n+1)+n}^2 + L_{(n+1)-n}^2 \\ &= 5F_{2n+1}^2 + L_1^2 \\ &= 5F_{2n+1}^2 + 1 \end{aligned}$$

Thus

$$L_{2n+2}L_{2n} - 5F_{2n+1}^2 = 1.$$

Also solved by J. L. Brown, Jr.; J. A. H. Hunter; Douglas Lind, Kathleen Marafino, Gary C. McDonald, C. B. A. Peck, Benjamin Sharpe, M. N. S. Swamy, Howard L. Walton, John Wessner, Kathleen M. Wickett, David Zeitlin, Charles Ziegenfus, and the proposer. Also by David Klarner.

MODULO THREE

B-61 *Proposed by J. A. H. Hunter, Toronto, Ontario*

Define a sequence $U_1, U_2, \ldots$ by $U_1 = 3$ and

$$U_n = U_{n-1} + n^2 + n + 1 \quad \text{for } n > 1 \;.$$

Prove that $U_n \equiv 0 \pmod n$ if $n \not\equiv 0 \pmod 3$.

Solution by John Wessner, Melbourne, Florida

An alternative representation for U_n is

$$U_n = \sum_{k=1}^{n} (k^2+k+1).$$

Upon expanding the individual sums involved we obtain

$$U_n = [n(2n+1)(n+1)/6] + [n(n+1)/2] + n = (n/3)\ [(n+2)(n+1)+3]\ .$$

Hence, $U_n \equiv 0 (\mathrm{mod}\ n)$ if and only if $(n+1)(n+2) \equiv 0(\mathrm{mod}\ 3)$. This condition obtains if and only if $n \not\equiv 0(\mathrm{mod}\ 3)$.

Also solved by Robert J. Hursey, Jr., Douglas Lind, Gary C. McDonald, Robert McGee, C. B. A. Peck, Charles R. Wall, David Zeitlin, and the proposer.

UNIQUE SUM OF SQUARES

B-62 *Proposed by Brother U. Alfred, St. Mary's College, California*

Prove that a Fibonacci number with odd subscript cannot be represented as the sum of squares of two Fibonacci numbers in more than one way.

Solution by J. L. Brown, Jr., Pennsylvania State University, State College, Pa.

From the identity $F_{2n+1} = F_n^2 + F_{n+1}^2$, $(n \geq 1)$ it follows that $F_{2n+1} < (F_n + F_{n+1})^2 = F_{n+2}^2$. Therefore, any representation, $F_{2n+1} = F_k^2 + F_m^2$ $(k \leq m)$ must have both k and $m \leq n+1$. Then $k \geq n$ (otherwise $F_k^2 + F_m^2 < F_n^2 + F_{n+1}^2 = F_{2n+1}$ for $k > 2$).

Also solved by Douglas Lind, Joseph A. Orjechouski and Robert McGee (jointly), C. B. A. Peck, and the proposer.

AN ISOSCELES TRIANGLE

B-63 *An old problem whose source is unknown, suggested by Sidney Kravitz, Dover, New Jersey.*

In ΔABC let sides AB and AC be equal. Let there be a point D on side AB such that AD = CD = BC. Show that

$$2\cos \sphericalangle A = AB/BC = (1 + \sqrt{5})/2\ ,$$

the golden mean.

Solution by John Wessner, Melbourne, Florida

By inspection of the figure and the law of cosines

$$AD^2 = CD^2 + AC^2 - 2CD \cdot AC \cos \sphericalangle A.$$

Since AD = CD = BC and AB = AC, it follows immediately that

$$2 \cos \measuredangle A = AC/CD = AB/BC.$$

The second result comes from the fact that

$$\measuredangle B = \measuredangle BDC = \measuredangle A + \measuredangle DCA = 2 \measuredangle A$$

and hence

$$\measuredangle A = 36^\circ \text{ and } 2 \cos A = (1 + \sqrt{5})/2.$$

(See N. N. Vorobyov: The Fibonacci Numbers (New York, (1961) p. 56.)

Also solved by Herta Taussig Freitag, Cheryl Hendrix, Kathleen Marafino, and Carol Barrington (jointly), J. A. H. Hunter, Douglas Lind, James Leissner, C. B. A. Peck, Kathleen M. Wickett, and the proposer.

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Continued from page 234.

5. H. Winthrop, "The Mathematics Of The Round Robin," Mathematics Magazine (In Press).
6. H. Winthrop, "A Mathematical Model For The Study Of The Propagation Of Novel Social Behavior," Indian Sociological Bulletin, July 1965, Vol. II. (In Press)
7. H. Winthrop, "Some Generalizations Of The Dying Rabbit Problem," (In Preparation).
8. N. N. Vorob'ev, Fibonacci Numbers, Blaisdell Publishing Company, New York, 1961.

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Brother U. Alfred has just completed a new booklet entitled: Introduction to Fibonacci Discovery. This booklet for teachers, researchers, and bright students can be secured for $1.50 each or 4 copies for $5.00 from Brother U. Alfred, St. Mary's College, Calif.