FIBONACCIOUS FACTORS

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1. INTRODUCTION

In earlier issues of the Quarterly there have been shown and proven answers to the following questions about the basic series (1, 1, 2, 3, 5 ---).

(1) By what primes are the various terms, U_n, divisible?

(2) At what points do various primes first appear as factors?

(3) At what periods do they reappear?

In this paper we deal with answers to the same questions as to the general series (a, b, a + b, a + 2b, 2a + 3b ---).

2. PERIODS OF REAPPEARANCE ARE THE SAME

Our task is simplified if we answer the last question first:

> If k is the period at which a prime repeats its zero residues in the basic series, k is also the period of zero residues in any general series.

Suppose that a prime first divides the nth term of a given series (a, b, a + b ---) and let the (n-1)th term be c. Then modulo p, (which we hereafter abbreviate to "[p") the series runs in this neighborhood as c, 0, c, c, 2c, 3c etc. The terms after the zero are those of the basic series each multiplied by c. Now if $x \not\equiv o$ [p, so also $cx \not\equiv o$ if $c \not\equiv o$ [p. Again, if $x \equiv o$ [p, so also $cx \equiv o$ [p. This means that in the two series (1, 1, 2, ---) and (c, c, 2c ---) the zeros appear at the same terms

3. SUMMARY OF PREVIOUS RESULTS AS TO FIRST APPEARANCES

(1) There are some terms of the basic series divisible by any prime one may choose.

(2) The term $U_{abc} \ldots$ is divisible by U_a, U_b U_c ... E.G.

$U_{12} = 144$

is divisible by

U $U_2 = 1$
$U_3 = 2$
$U_4 = 3$
$U_6 = 8$

(3) Such a term U_n, for which n is composite, may also have other factors, called "primitive prime divisors;" and the general form of these primes is determined by the following rules (but their identity must be found by trial and error).

(A) If n is odd; p is of the form $2\,kn \pm 1$

(B) If $n = 2\,(2\,r + 1)$; p is of the form $nk \pm 1$

(C) If $n = 2^m\,(2\,r + 1)$; p is of the form $nk/2 - 1$

Examples are listed in the February 1963 Quarterly at pp. 44-45.

(4) The fact that n is prime does not imply that U_n is prime. E. g., U U 19 = 4181 = 37 x 113; even though 19 is prime. However, the converse is true: If U_n is prime, so also is n.

(5) The even prime, 2, is a factor of every third term of the series; and the odd prime 5 is a factor of every 5th term.

(6) All other odd primes are of the forms ± 1 and ± 3 [10. They appear and reappear as factors according to the following rules:

(a) If $p \equiv \pm 1$ [10, it will first appear when the n of $U_n = \frac{p-1}{d}$, d being some positive integer; and will reappear every nth term thereafter;

(b) If $p \equiv \pm 3$ [10, it will first appear when $n = \frac{p+1}{d}$, and every nth term thereafter, d again being some positive integer. E. g.,

3	divides	U_4	and	every	4th	term	thereafter
7	"	U_8	"	"	8th	"	"
11	"	U_{10}	"	"	10th	"	"
13	"	U_7	"	"	7th	"	"
17	"	U_9	"	"	9th	"	"
19	"	U_{18}	"	"	18th	"	"

(c) The rules for determining the divisor, d, of p ± 1 in (6) have not yet been given. Examination of the primes less than 80 give d = 1, 2 or 4 in all cases except 47, where it is 3. However, in the range from p = 2,000 to 3,000, given in the February 1963 issue at pp. 36-40, d has values ranging from 1 to 78.

(7) Nothing has thus far been said about the appearances and periods of composite factors, ab $(a \neq b)$, nor factors which are powers, p^c.

4. NEW ANSWERS TO THE QUESTION OF FIRST APPEARANCES

(1) "By what factors are the terms of the general series (a, b, a + b, a + 2b, + 3b . . .) divisible?"

It can be shown that if A, B and C denote any three successive terms in this series, then $B^2 - AC = \pm$ a constant, no matter which three terms are chosen, and no matter what the values of A and B (the first two terms).

Specifically, work on the first few terms of the general series shows what this constant must be

$$b^2 - a(a + b) = b^2 - ab - a^2$$

$$\text{or} \quad (a + b)^2 - b(a + 2b) = a^2 + 2ab + b^2 \quad - ab - ab^2$$

$$= - b^2 + ab + a^2$$

$$= -(b^2 - ab - a^2)$$

How can we make use of this constancy of $B^2 - AC$ to determine the possibility of a given prime, p, as a factor of some term in the general series? By changing the equation to a congruence [p. If any term, C, of the series is divisible by p; then C and its two immediate predecessors must satisfy the congruence

$$B^2 - AC \equiv \pm (b^2 - ab - a^2) \ [p$$

But we are assuming $C \equiv 0$ [p. This eliminates the term - AC. Hence we must have $B^2 \equiv \pm (b^2 - ab - a^2)$ [p.

In other words, once we know the first two terms, a and b of a general series; we know that the only possible factors for terms of the series are those for which $\pm (b^2 - ab - a^2)$ is a quadratic residue. Primes of which this is not true cannot be the modulus in the congruence

$$B^2 \equiv \pm (b^2 - ab - a^2) \ [p.$$

However, it does not follow from the necessity of this condition that it is also sufficient. E.g., 1, 4, 5 ... is never divisible by 89. Nevertheless, Brother Alfred has shown that there are some primes which are factors of all Fibonaccious series.

(2) We can no longer say that U_{abc} is divisible by U_a, U_b and U_c, as a single example will show. Consider 3, 7, 10, 17, 27, 44, 71, 115, 186, 301. $U_{10} = 301$ is divisible by $U_2 = 7$, but not by $U_5 = 27$.

(3) Neither can we say of a general series that if U_n is prime, so too is n. Vide 2, 5, 7, 12, 19, 31 ... for which U_6 is prime but 6 is not.

(4) (a) Nor do we have in the general series a set of primitive prime factors, in view of (2) above.

(b) Thus we are fairly limited, as to rules for the forms of certain, possible or impossible prime factors of the general series. We make here only two observations:

(i) For primes of the form $p = 4k + 3$, either a or -a is a residue for any value of a. Hence these primes are possible, but not necessarily certain factors of any general series.

(ii) On the other hand, for primes of the form $p = 4k + 1$, there can be values of a for which neither a nor -a is a residue. E.g., neither 2 or -2 is a residue [5; and neither ± 2 nor ± 5 nor ± 6 are residues [13. Hence these primes are impossible factors of general series for which the initial terms are correctly chosen.

E.g., no terms of the series 1, 63, 64, 127 are ever divisible by 5, 11, 13 or 17, since $\pm(63^2 - 64) = \pm(3969-64) = \pm 3905$ is a non-residue of each of these primes.

Hence let us put aside for the moment the more particular rules of forms of factors of the general series, and turn to the place of first appearance of possible factors. The intervals of reappearance are as in the basic series.

(5) First let us review 2 and 5. If any series is reduced 2, we have only four patterns, depending on choice of initial terms

1, 1, 0, 1, 1, 0, 1, 1, 0, ...
0, 0, 0, 0, 0, 0, 0, 0 ......
1, 0, 1, 1, 0, 1, 1, 0 ....
0, 1, 1, 0 ......

That is to say: one of the first three terms must be even; and thereafter either all or every 3rd term is even.

For 5, the situation is a little more complex. Actual computation of first appearances for the various combinations of remainders of the first two terms enables us to make the following table:

If the second term has a remainder of

and the first a remainder of	0	1	2	3	4	[5
0	1	1	1	1	1	
1	2	5	4	N	3	
2	2	N	5	3	4	
3	2	4	3	5	N	
4	2	3	N	4	5	

the entries show the number of the smallest term divisible by 5, where N signifies "none." Thus we see that 5 may first appear as a factor of any term from the 1st to the 5th, or be suppressed entirely; by proper choice of first terms. However, as the reader can easily verify, if 5 appears once as a factor, it reappears in every 5th term thereafter.

(6) Now, as before, let us turn from these two special cases of 2 (the only even prime) and 5 (the only one $\equiv 5$ [10) and consider the remaining ones of the forms ± 1 and ± 3 [10. We make the following conjectures:

(a) By proper choice of initial terms we can make any such prime, p, first appear as a factor of any term whose number (rank)$<$p; or, if p is of the form $4k + 1$, we can suppress it altogether.

(b) If such a prime appears at all, it will reappear at the same interval as in the basic series.

To test these conjectures, let us make tables, as for 5, for 7 and 11.

	0	1	2	3	4	5	6
0	1	1	1	1	1	1	1
1	2	8	7	4	5	6	3
2	2	5	8	6	7	3	4
3	2	6	4	8	3	5	7
4	2	7	5	3	8	4	6
5	2	4	3	7	6	8	5
6	2	3	6	5	4	7	8

Note the absence of N's; since 7 is always a factor of some terms of any general series.

For 11: b,

Second term

	0	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1	1	1
1	2	10	9	5	N	4	6	8	N	7	3
2	2	6	10	8	9	N	5	7	N	3	4
3	2	N	N	10	4	7	9	6	3	5	8
4	2	5	6	7	10	N	8	3	9	4	N
5	2	7	8	4	5	10	3	N	6	N	9
6	2	9	N	6	N	3	10	5	4	8	7
7	2	N	4	9	3	8	N	10	7	6	5
8	2	8	5	3	6	9	7	4	10	N	N
9	2	4	3	N	7	5	N	9	8	10	6
10	2	3	7	N	8	6	4	N	5	9	10

Observing these three tables, we see the following common features:

(i) The top line is always all 1's;

(ii) The left column is always all 2's, except for the top entry.

(iii) One diagonal is all 3's.

(iv) The other diagonal is all k's (where k will be seen to be the constant of reappearance, in this case 10), except for the upper left corner.

(v) The nth line (except the top) is line 1 "spaced out" at intervals of m from the 3.

(vi) Hence only line 1 need be computed.

Some of the features are obvious:

(i) The top line of 1's mean only that a (in the series a, b, a + b ...) ≡ o [p. Hence the first zero is at the first term.

(ii) The left column of 2's is similarly explicable. The exception of 1 at the top left corner is because both a and b ≡ o, and the earlier of the two is a, the 1st term.

(iii) The diagonal of 3's is due to their representing series in which the first two terms are a, p-a, p. The a's vary; but p in the 3rd term does not.

(iv) The identities in the other diagonal represent general series of which the first two terms are both a (2, 2, 4 ..., 3, 3, 6 ..., 4, 4, 8 ...). The terms of each of these series are those of the basic (1, 1, ...) each multiplied by a. Consequently if any term in the basic series gave a remainder [p it would also give a remainder (usually different) when multiplied by a constant. On the other hand, if the nth term, U_n, of the basic series ≡ o [p; so also a U_n ≡ o [p. That is to say, the earliest zero remainder in (a, a, 2a ...) occurs at the same term, regardless of the value of a.

(v) The "spacing out" of Line 1 to get the entries in Line n of the table is explicable similarly. If x ≡ o [p so also kx ≡ o [p while if x ≢ o [p so also kx ≢ o [p, in the first case for any value of k, and in the second so long as k ≢ o [p.

This means that the occurrence of zeros in any series (a, b, a + b ...) is unchanged if each term in the series is multiplied by the same constant, k ≢ o [p. In other words, while non-zero remainders may vary, p will occur as a factor of precisely the same terms in series (1, 2, 3, 5 ...), (2, 4, 6, 10 ...), (3, 6, 9, 15 ...) etc. Hence the entries in line 1 and col. 2, line 2 and col. 4, line 3 and col. 6 of the table must be the same; and similar reasoning shows how the rest of the spacing out follows the same pattern.

(vi) Finally we must consider line "1" of the table. To fill it out the hard and obvious way requires us to run out, reduced [p, the various series (1, 2, 3, 5 ...), (1, 3, 4, 7 ...), (1, 4, 5, 9 ...) until we reach a zero in each; and then make corresponding entries in line 1. This done, spacing out as per (v) will complete the table.

An alternative, or a cross-check can be made as follows: Suppose we run out the basic series for a prime we have not yet considered, 13. The series reduced [13 to the first zero is 1, 1, 2, 3, 5, 8, 0.

Attached is a table partially filled in, with the invariable 1st row of 1's, left column of 2's, diagonal of 3's, and diagonal of 7's (the zero period of the basic series). There are other entries, which we now explain.

For [13

Remainder of Second Term (b)

Remainder of First Term (a)	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1	1	1	1	1	1	1	1	1	1	1	1	1
1	2	7	6										3
2	2		7	5								3	4
3	2			7		4					3	5	
4	2				7					3			
5	2					7			3		6		
6	2						7	3					
7	2						3	7					
8	2			6		3			7				
9	2				3					7			
10	2		5	3					4		7		
11	2	4	3								5	7	
12	2	3										6	7

1	1	2	3	5	8	0	marked ///
8	8	3	11	1	12	0	" \\\
12	12	11	10	8	5	0	" ≡
5	5	10	2	12	1	0	" ||||

The entry in (1, 1) is 7; because we have just seen that 7 is the zero-period of the basic series. There is similarly a 6 in the square (1, 2) because after a look at the basic series, we see that if we start a new series with first terms 1, 2, instead of 1, 1; we arrive at 0 after 6 terms instead of 7. In fact, as the 7 and 11 tables have illustrated already, the entry in square (1, 2) of the table is always k-1, where k is the number of the first zero term in the basic series. Similarly the entry in the square (2, 3) is always k-2; and in the square (3, 5) it is k-3; etc.; because as we select later and later pairs of terms in the basic series to start new series, we reduce one by one the number of the first term in which zero appears. Hence we can, without further computation than the basic series reduced [p, fill in a number of entries on various lines of the zero appearance table (see the attached figure for 13).

Moreover, we can use these entries, with a little more trial and error, to work back to values in line 1 of the table. For example, let us again look at the 13 table. The period of zero-appearances being 7 (as we have seen from the basic series) and 3, 5 being the 4th and 5th terms in the basic series, we know that 0 appears at the (7-3)th term in a new series (3, 5, 8, 0 ...). Suppose we multiply the new series, term by term, by such a factor (9) as makes a still newer series with the first term 1.

We have	3 x 9,	5 x 9,	8 x 9,	Ox 9	---	[13
or	27 ,	45,	72,	0	---	[13
or	1 ,	6,	7,	0	---	[13

Hence from the entry of 4 in square (3, 5) we can check the same entry in (1, 6); both must be and are 4.

Here we note an interesting point. Still working with modulus 13, we have the basic series

	1, 1, 2, 3, 5, 8, 8,	first zero 7
from which we get	1, 2, 3, 5, 8, 0,	zero 6
	2, 3, 5, 8, 0	zero 5
	3, 5, 8, 0	zero 4
	5, 8	zero 3

We have found the entry in the table (first zero) for 3, 5 was the same as for (1, 6). Similarly we have seen that (1, 2) is simply 1 less than (1, 1). Again (2, 3) ≡ (14, 21) ≡ (1, 8); and (5, 8) ≡ (40, 64) ≡ (1, 12). However, this gives us entries in line 1 only for claims 1, 2, 6, 8 and 12. We have no data for the remaining columns, i.e. for series beginning (1, 3) (1, 4) (1, 5) (1, 7) (1, 9) (1, 10) and (1, 11).

One might at first imagine that these deficiencies were due to the fact that we had only run our basic series out to the first zero, instead of continuing beyond this restricted period to the full period, when not only zero but all remainders [13 repeat: 1, 1, 2, 3, 5, 8, 0, 8, 8, 3, 11, 1, 12, 0, 12, 12, 11, 10, 8, 5, 0, 5, 5, 10, 2, 12, 1, 0. However, the reader will find that the new entries in squares (8, 8) (8, 3) (3, 11) etc. still "run back" to the same set of 5 entries on line 1.

There are no entries on line 1 in columns 3, 4, 5, 7, 9, 10 and 11; because series with first terms 1 and second terms 3, 4, 5, 7, 9, 10 and 11 have no terms divisible by 13! Recall our test, of whether p could be a factor of a series beginning (1, b, 1 + b), i.e., is $\pm b^2 - b - 1$ a residue of p? It will be found that

$$\pm(3^2 - 3 - 1) \equiv 5 \text{ or } 8$$

$$\pm(4^2 - 4 - 1) \equiv 11 \text{ or } 2$$

$$\pm(5^2 - 5 - 1) = \pm 19 \equiv 6 \text{ or } 7$$

$$\pm(7^2 - 7 - 1) = \pm 41 \equiv 2 \text{ or } 11$$

$$\pm(9^2 - 9 - 1) = \pm 71 \equiv 6 \text{ or } 7$$

$$\pm(10^2 - 10 - 1) = \pm 89 \equiv 11 \text{ or } 2$$

$$\pm(11^2 - 11 - 1) = \pm 109 \equiv 5 \text{ or } 8$$

are none of them residues [13.

Consequently there must be entries of N (for "never") in each of Columns 3, 4, 5, 7, 9, 10 and 11 of Line 1.

TO SUMMARIZE as to the appearances of p as a factor of terms in a general series (a, b, a + b).

If p is prime

(i) It will never appear unless $\pm(b^2 - ab - a^2)$ is a residue of p.

(ii) If it can appear per (i), and does so, it will reappear at the same interval as in the basic series.

(iii) To determine the place of first appearance there is no simpler method known to the writer than to reduce a and b [p and then run the series out to the first zero. However, this can be quite a bit simpler than running out the series, itself. E.g., what, if any, terms are divisible by 19 in the series 119, 231, 350, 581? Note

$119 \equiv 5 \quad [19 \qquad 231 \equiv 3 \quad [19$

Hence the first 3 terms $\equiv$ 5, 3, 8 and $3^2 - 5.8 = -31 \equiv 112 \equiv 7$, a residue; so that 19 is a possible factor, then we have 5, 3, 8, 11, 0. I.e., the 5th term 931 is so divisible. Moreover, since the zero period of the basic series is 18; this is also the period in our given series; and the 23rd, 41st and every 18th term thereafter is divisible by 19.

If p is composite, the rules for zero appearances can be derived from the rules of its prime factors in a manner easily illustrated by two examples:

(1) What, if any, terms are divisible by 143 in the series

1, 6, 7, 13 --- ?

Since 143 = 11 x 13 we first check possibility of both primes as factors

$$6^2 - 6 - 1 = 29 \equiv 7 \quad [11 \text{ and } 3 \quad [13$$

$-7 \equiv 4$ is a residue of 11; and 3 is a residue of 13.

Hence both primes are possible factors

Moreover, it can easily be found that zero [11 appears at the 6th term with a period of 10; while zero [13 appears at the 4th term with a period of 7.

Hence the number n, of the first term divisible by 143 must satisfy the congruences.

$$n \equiv 6 \quad [10$$

$$\text{and } \; n \equiv 4 \quad [13$$

The minimum solution is 56. Hence the 56th term is the smallest divisible as required by 143.

(2) On the other hand, there are cases in which, while there may be terms of a series divisible by each of two (or more) primes, there may be none divisible by both (or all). Consider

1, 7, 8, 15, 23

As the reader can check, the 4th term and every 5th thereafter is divisible by 5; while the 8th term (99) and every 10th thereafter is divisible by 11. However, there is no term divisible by 55. This is due to the fact that there is no solution to the simultaneous congruences

$$n \equiv 4 \ [5 \quad \text{(a number ending in 4 or 9)}$$

$$n \equiv 8 \ [10 \quad \text{(a number ending in 8)}$$

No number satisfies both conditions.

Thus there is no fixed and simple test for divisibility of a general series by a composite number. One must determine for each prime factor of the composite modulus, (i) the term at which it first appears and (ii) the period at which it reappears thereafter. Then one must test the congruences expressing these two conditions for each prime in the composite modulus; and either solve them or find them to be insoluble.

To complete this analysis would require attack on the problem of zero appearances in both the basic and general series for moduli which are powers of primes, p^c. However, this discussion is postponed pending publication of a proof by J. H. E. Cohn that in the basic series no terms are exact squares, except U_1, U_2 and U_7.

Beyond this we offer only these Conjectures:

In the basic series

(i) If the k^{th} term is the first one divisible by p, then the choice of first two terms, and will not be greater than the (p^{c-1})th term.

(ii) There will be no first appearance, if the first terms are chosen so that $\pm(b^2 - ba - a^2)$ are nonresidues $[p^c$.

(iii) If there is a first appearance, there will be reappearances at the same period as in the basic series.

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