TWO FIBONACCI CONJECTURES

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Consider the problem of solving, in positive integers, the following Diophantine equation (suggested by the Editor):

$$F_n x + F_{n+1} y = x^2 + y^2. \tag{1}$$

First let us note that (1) always has the trivial solution (F_n, F_{n+1}), i.e., $x = F_n$, $y = F_{n+1}$. Does (1) ever have a non-trivial solution? If n is fixed, we know from analytic geometry that there are at most a finite number of solutions. However, we shall soon see that for infinitely many n (1) has at least two non-trivial solutions.

Theorem 1. If $n > 1$ and $n \equiv 1 \pmod 3$, then

$$\left(\frac{F_{n+2}}{2}, \frac{F_{n+2}}{2}\right)$$

and

$$\left(\frac{F_{n+2}}{2}, \frac{F_{n-1}}{2}\right)$$

are non-trivial solutions of (1).

Proof. Since $n \equiv 1 \pmod 3$, $F_{n-1} \equiv F_{n+2} \equiv 0 \pmod 2$ which guarantees that the quotients involved are indeed integers. One may immediately verify that they satisfy (1).

Theorem 2. If (x_o, y_o) is a solution of (1), than $u = 2x_o - F_n$, $v = 2y_o - F_{n+1}$ is a solution of

$$u^2 + v^2 = F_{2n+1}. \tag{2}$$

Proof. This is an immediate consequence of the identity (Lucas, 1876)

$$F_n^2 + F_{n+1}^2 = F_{2n+1}.$$

If $u = u_o$ and $v = v_o$ is a solution of (2) with $(u_o, v_o) \neq (F_n, F_{n+1})$ (or any of the other 7 solutions of (2) obtained by changing signs or interchanging F_n and F_{n+1}) we shall call (u_o, v_o) a non-trivial solution of (2).

Theorem 3. If $n \not\equiv 1 \pmod 3$, then (1) has a non-trivial solution if and only if (2) has a non-trivial solution.

Proof.

(a) If (x_o, y_o) is a non-trivial solution of (1), then by Theorem 2, $u = 2x_o - F_n$, $v = 2y_o - F_{n+1}$ is a solution of (2). If $2x_o - F_n = \pm F_n$, then $x_o = F_n$ (and hence $y_o = F_{n+1}$ or 0) or $x_o = 0$, a contradiction. If $2x_o - F_n = \pm F_{n+1}$, $x_o = F_{n+2}/2$ or $x_o < 0$ which is impossible since F_{n+2} is odd and we are considering only positive solutions of (1).

(b) Let us assume $u_o > 0$, $v_o > 0$ is a non-trivial solution of (2). F_w is even if and only if $w \equiv 0 \pmod 3$; thus by hypothesis F_{2n+1} is odd. But $F_{2n+1} = u_o^2 + v_o^2$, hence u_o and v_o must be of different parity. Moreover, for the same reason F_n and F_{n+1} must also be of different parity. Thus (interchanging names if necessary) we may be sure that

$$\left(\frac{u_o + F_n}{2}, \frac{v_o + F_{n+1}}{2}\right)$$

is an integral solution of (1). If

$$\frac{u_o + F_n}{2} = F_n$$

we would, as before, get a contradiction.

The reader is invited to show that the <u>number</u> of non-trivial solutions of (1) is always even.

Now the problem of representing a number as the sum of two squares has received considerable attention. The following result, known to Fermat and others was proved by Euler:

If $N = bc^2 > 0$, where b is square-free, then N is representable as the sum of two squares if and only if b has no prime factors of the form $4k + 3$.

Theorems on the number of such representations can be found in virtually every introductory text on number theory.

Thus by Theorem 3 if $n \not\equiv 1 \pmod 3$ and F_{2n+1} is a prime of form $4k+1$, the only solution of (1) in positive integers is (F_n, F_{n+1}) since every prime of the form $4k+1$ is the sum of two squares in essentially only one way.

It is interesting to note that the pertinent identity

$$(a^2 + b^2)(c^2 + d^2) = (ac \pm bd)^2 + (ad \mp bc)^2$$

was given by Fibonacci in his Liber Abaci of 1202. This can be used to expedite numerical investigations. However, one needs to beware (or at least be aware) of such accidents as the following:

Let $n > o$, $n \equiv 2 \pmod 3$;

(i) If $n < 32$ (and $n \neq 17$), then $2n+1$ is a prime.

(ii) If $n < 17$, both $2n+1$ and F_{2n+1} are primes!

Another useful result is

Theorem 4. $F_{2n+1} \equiv 1, 2$, or $5 \pmod 8$.

Proof. We shall use the identity $F_{2n+1} = F_n^2 + F_{n+1}^2$. If g is odd, then $g^2 \equiv 1 \pmod 8$. Thus if F_n and F_{n+1} are both odd, $F_{2n+1} \equiv 2 \pmod 8$. Since two consecutive Fibonacci numbers are relatively prime, the only remaining possibility is that F_n and F_{n+1} are of different parity; in this case we get $F_{2n+1} \equiv 1$ or $5 \pmod 8$.

The reader may prove that in general $F_n \not\equiv 4 \pmod 8$.

Finally, this problem suggests the following conjectures:

Conjecture 1. There are infinitely many values of n for which (1) has only a trivial solution.

Conjecture 2. F_{2n+1} is never divisible by a prime of the form $4k+3$.

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