ADVANCED PROBLEMS AND SOLUTIONS

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Send all communications concerning Advanced Problems and Solutions to Verner E. Hoggatt, Jr., Mathematics Department, San Jose State College, San Jose, California. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-70 *Proposed by C. A. Church, Jr., West Virginia University, Morgantown, West, Va.*

For $n = 2m$ show that the total number of k-combinations of the first n natural numbers such that no two elements i and $i+2$ appear together in the same selection is F^2_{m+2}, and if $n = 2m+1$, the total is $F_{m+2}F_{m+3}$.

H-71 *Proposed by John L. Brown, Jr., Penn State University, State College, Pennsylvania*

Show

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} 2^{k-1} L_k = 5^n$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} 2^{k-1} F_k = 0 \quad .$$

H-72 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Let $u_n = F_{nk}$, where F_m is the mth Fibonacci number, and k is any positive integer; and let

$$\begin{bmatrix} m \\ 0 \end{bmatrix} = \begin{bmatrix} m \\ m \end{bmatrix} = 1, \quad \begin{bmatrix} m \\ n \end{bmatrix} = \frac{u_m \cdots u_1}{u_n u_{n-1} \cdots u_1 \, u_{m-n} u_{m-n-1} \cdots u_1} \quad ,$$

then show

$$2\begin{bmatrix} m \\ n \end{bmatrix} = L_{nk}\begin{bmatrix} m-1 \\ n \end{bmatrix} + L_{(m-n)k}\begin{bmatrix} m-1 \\ n-1 \end{bmatrix} .$$

H-73. *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Let $f_o(x) = 0$, $f_1(x) = 1$, and

$$f_{n+2}(x) = xf_{n+1}(x) + f_n(x) \qquad n \geq 0$$

and let $b_n(x)$ and $B_n(x)$ be the polynomials in H-69, show

$$f_{2n+2}(x) = x B_n(x^2) \quad ,$$

and

$$f_{2n+1}(x) = b_n(x^2) \quad ,$$

thus there is an intimate relationship between the Fibonacci polynomials, $f_n(x)$, and the Morgan-Voyce polynomials $b_n(x)$ and $B_n(x)$.

H-74 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*

Let f(n) denote the number of positive Fibonacci numbers not greater than a specified integer n. Show that for $n > 1$

$$f(n) = \left[K \ln(n \sqrt{5} + \tfrac{1}{2})\right] ,$$

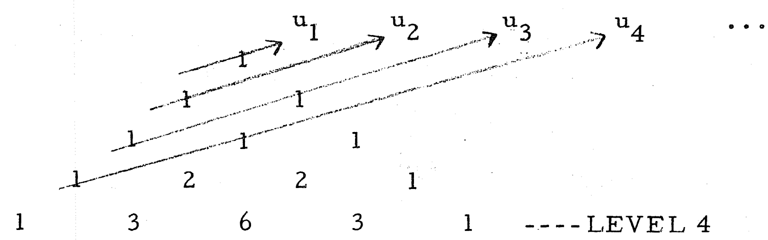
where $[x]$ denotes greatest integer not exceeding x, and K is a constant nearly equal to 2.078086943. (See H. W. Gould's Non-Fibonacci Numbers, Oct., 1965, FQJ).

H-75 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*

Show that the number of sets of distinct integers with one element n, all other elements less than n and not less than k, and such that no two consecutive integers appear in the set is F_{n-k+1}.

H-76 *Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California*

It is well known that the Fibonacci numbers are sums of the rising diagonals of Pascal's triangle. Find a recurrence relation for the rising diagonals for the Fibonomial triangle:



$$2\begin{bmatrix}m\\n\end{bmatrix} = L_n\begin{bmatrix}m-1\\n\end{bmatrix} + L_{m-n}\begin{bmatrix}m-1\\n-1\end{bmatrix} \qquad \begin{bmatrix}m\\o\end{bmatrix} = \begin{bmatrix}m\\m\end{bmatrix} = 1$$

$u_1 = 1$, $u_2 = 1$, $u_3 = 2$, $u_4 = 2$, $u_5 = 4$, $u_6 = 6$ etc. See H-63 April 1965 FQJ p. 116 and H-72 this issue.

H-77 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Show

$$\sum_{j=0}^{2n+1} \binom{2n+1}{j} F_{2k+2j+1} = 5^n L_{2n+2k+2}$$

for all integers k. Set $k = -(n+1)$ and derive

$$\sum_{j=0}^{n} \binom{2n+1}{n-j} F_{2j+1} = 5^n ,$$

a result* of S. G. Guba Problem #174 Issue #4 July-August 1965 p. 73 of Matematika V Škole.

AN ALTERNATE FORM

H-49 *Proposed by C. R. Wall, Texas Christian University, Ft. Worth, Texas*

Show that, for $n > 0$,

$$2^n F_{n+1} = \sum_{m=0}^{n} \frac{5^{[m/2]}\, n^{(m)}}{m!}$$

*Reported by H. W. Gould.

where [x] denotes the integral part of x, and $x^{(n)} = x(x-1)\ldots(x-n+1)$.

Solution by Douglas Lind, University of Virginia, Charlottesville, Va.

Solution: We first note $\binom{n}{m} = n^{(m)}/m!$ Horner ("Fibonacci and Pascal," Fibonacci Quarterly, Vol. 2, No. 3, p. 228) has given equivalently

$$2^n F_{n+1} = \sum_{k=0}^{\left[\frac{n+1}{2}\right]} \binom{n+1}{2k+1} 5^k,$$

so that

$$2^n F_{n+1} = \sum_{k=0}^{\left[\frac{n+1}{2}\right]} \left\{\binom{n}{2k} + \binom{n}{2k+1}\right\} 5^k$$

$$= \sum_{m=0}^{n} 5^{[m/2]} \binom{n}{m},$$

the desired result.

OOPS!

H-26 was finally solved by Douglas Lind and the solution appeared in the last issue.

PROBLEMS AND PAPERS

H-46 *Proposed by F. D. Parker, SUNY at Buffalo, Buffalo, New York*

Prove

$$D_n = |a_{ij}| = (-1)^n K,$$

where $a_{ij} = F^4_{n+1+j-2}$ (i, j = 1, 2, 3, 4, 5) and find the value of K. This problem and its generalizations will be discussed in separate papers by D. Klarner and L. Carlitz to appear later in the Quarterly.

NON-HOMOGENEOUS FIBONACCI

H-48 *Proposed by J. A. H. Hunter, Toronto, Ontario, Canada*

Solve the non-homogeneous difference equation

$$C_{n+2} = C_{n+1} + C_n + m^n ,$$

where C_1 and C_2 are arbitrary and m is a fixed positive integer.

Solution by Raymond E. Whitney, Lock Haven State College Lock Haven, Pennsylvania

Using the standard technique of converting the difference equation to a differential equation with the transform

$$Y(t) = \sum_{o}^{\infty} C_i t^i / i! \qquad (C_o \equiv C_2 - C_1 - 1),$$

we obtain

$$Y''(t) = Y'(t) + Y(t) + e^{mt}$$

Thus

$$Y(t) = A_e \left[(1 + \sqrt{5})/2\right] t + B_e \left[(1 - \sqrt{5})/2\right] t + \left[1/(m^2 - m - 1)\right] e^{mt}.$$

Hence

$$C_n = Y^{(n)}(o)$$

$$= A \left[(1 + \sqrt{5})/2\right]^n + B \left[(1 - \sqrt{5})/2\right]^n + m^n/(m^2 - m - 1),$$

where A, B are determined via boundary conditions $\left[C_o, C_1\right]$.

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