A NOTE ON FIBONACCI SUBSEQUENCES

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The question has been raised, whether certain subsequences of the Fibonacci sequence

(1) $$F_0 = 0, \; F_1 = 1, \quad F_{n+1} = F_n + F_{n-1} ,$$

can themselves be obtained directly from a recurrence-relation.

First, consider a <u>periodic subsequence</u>, $P_n = F_{nq+r}$, of every q-th Fibonacci number, starting with F_r. It is known (see, e.g., D. Ruggles, Fibonacci Quarterly 1(1963)2:77) that

(2) $$F_{p+q} = L_q F_p + (-1)^{q-1} F_{p-q} ,$$

Putting $p = nq + r$ and substituting the appropriate P_n, we obtain the hoped-for relation,

(3) $$P_0 = F_r , \; P_1 = F_{q+r} , \; P_{n+1} = L_q P_n + (-1)^{q-1} P_{n-1} .$$

On the other hand, we may wish to consider the <u>complementary sequence</u> of those F_i which are not of the form P_n. If these are written Q_k, it is easy to see that, after an initial $(r - 1)$ terms, this sequence comes in cycles of $(q - 1)$ consecutive F_i, and that

$$Q_1 = F_1 , \; Q_2 = F_2 , \; \ldots , \; Q_{r-1} = F_{r-1} \; ; \; Q_r = F_{r+1} , \; \ldots ,$$
$$Q_{n(q-1)+r} = F_{nq+r+1} , \; \ldots , \; Q_{n(q-1)+r+q-2} = F_{nq+r+q-1} , \; \ldots$$

Thus, $Q_{k+1} = Q_k + Q_{k-1}$, except when a P_n intervenes. If $q = 2$, we have the special situation, that there is a P_n between each adjacent pair of Q_k, and the complementary sequence is itself periodic and satisfies the relation (3):

(4) $$Q_{k+1} = L_2 Q_k - Q_{k-1} = 3Q_k - Q_{k-1} .$$

if $q \geq 3$, at most one P_n can intervene between Q_{k-1} and Q_{k+1}. This occurs if $k = n(q-1)+r-1$, so that the remainder R_k when $(k-r+1)$

is divided by $(q - 1)$ is 0, when $Q_{k+1} = F_{nq+r} + Q_k = 2Q_k + Q_{k-1}$; and if $k = n(q-1)+r$, so that $R_k = 1$, when $Q_{k+1} = F_{nq+r} + Q_k = 2Q_k + Q_{k-1}$, and if $k = n(q-1)+r$, so that $R_k = 1$, when $Q_{k+1} = Q_k + F_{nq+r} = 2Qk - Q_{k-1}$.

If $q = 3$, R_k can only be 0 or 1, and we get the rather simple relation

$$Q_{k+1} = 2Q_k + (-1)^{R_k} Q_{k-1} = 2Q_k + (-1)^{k-r+1} Q_{k-1} ; \tag{5}$$

but if $q \geq 4$, the neatest formula I could find was to define

$$S_k = \max (2 + R_k - R_k^2 , 1), \quad T_k = \min (R_k , 2),$$

when

$$Q_{k+1} = S_k Q_k + (-1)^{T_k} Q_{k-1} . \tag{6}$$

Alternatively, in terms of Kronecker's δ,

$$Q_{k+1} = \left\{1 + \delta_{OR_k} + \delta_{1R_k}\right\} Q_k + \left\{1 - 2\delta_{1R_k}\right\} Q_{k-1} . \tag{7}$$

An investigation of subsequences of the forms $X_n = F_{n^2}$ and $X_n = F_{2^n}$, for example, strongly suggests that only periodic sequences of the form P_n yield linear recurrence-relations with constant coefficients.

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