POWER IDENTITIES FOR SEQUENCES DEFINED BY $W_{n+2} = dW_{n+1} - cW_n$

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1. INTRODUCTION

Let W_o, W_1, $c \neq 0$, and $d \neq 0$ be arbitrary real numbers, and define

(1.1) $W_{n+2} = dW_{n+1} - cW_n$, $d^2 - 4c \neq 0$, $(n = 0, 1, \ldots)$,

(1.2) $Z_n = (\alpha^n - \beta^n)/(\alpha - \beta)$ $(n = 0, 1, \ldots)$,

(1.3) $V_n = \alpha^n + \beta^n$ $(n = 0, 1, \ldots)$,

where $\alpha \neq \beta$ are roots of $y^2 - dy + c = 0$. We shall define

(1.4) $W_{-n} = (W_o V_n - W_n)/c^n$ $(n = 1, 2, \ldots)$.

If $W_o = 0$ and $W_1 = 1$, then $W_n \equiv Z_n$, $n = 0, 1, \ldots$; and if $W_o = 2$ and $W_1 = d$, then $W_n \equiv V_n$, $n = 0, 1, \ldots$. The phrase, Lucas functions (of n) is often applied to Z_n and V_n, which may also be expressed in terms of Chebyshev polynomials (see (5.1) and (5.2)).

In this paper, general results (see section 3) have been obtained that yield new even power identities (Theorem 1) for sequences defined by (1.1). An additional result, Theorem 2, which contains Theorem 1 as a special case, yields identities whose typical term is the product of an even number of arbitrary terms taken from a given sequence defined by (1.1). Particular applications will be given for Fibonacci sequences and Chebyshev polynomials.

2. PRELIMINARIES

We shall need the following result:

Lemma 1. Let W_o, W_1, $c \neq 0$, and $d \neq 0$ be arbitrary real numbers, and let W_n, $n = 0, 1, \ldots$, satisfy (1.1). Let $m, p = 1, 2, \ldots$, and define

(2.1) $Q(n, p, m, i_1, \ldots, i_p) \equiv \prod_{s=1}^{p} W_{mn+i_s} = Q_n$ $(n = 0, 1, \ldots)$, where i_s, $s = 1, 2, \ldots, p$, are positive integers or zero. Then Q_n satisfies a homogeneous, linear difference equation of order $p+1$ with real, constant coefficients whose characteristic equation is $g(y) = 0$, where

$$(2.2)\quad g(y) \equiv \begin{cases} \prod_{j=0}^{(p-1)/2} (y^2 - c^{mj} V_{m(p-2j)} y + c^{mp}) & (p = 1, 3, 5, \ldots); \\ (y - c^{pm/2}) \prod_{j=0}^{(p-2)/2} (y^2 - c^{mj} V_{m(p-2j)} y + c^{mp}) & (p = 2, 4, 6, \ldots). \end{cases}$$

<u>Proof.</u> Let A, B, and C_s, $s = 0, 1, \ldots, p$, denote arbitrary constants. If $\alpha \neq \beta$ denote the roots of $y^2 - dy + c = 0$, then

$$W_n = A\alpha^n + B\beta^n$$

and

$$W_{mn+i_s} = A\alpha^{i_s}\alpha^{mn} + B\beta^{i_s}\beta^{mn}.$$

Observing that

$$Q_n = \sum_{s=0}^{p} C_s (\alpha^{m(p-s)}\beta^{ms})^n, \quad n = 0, 1, \ldots,$$

we can now conclude that Q_n satisfies a homogeneous, linear difference equation of order $p+1$ with real, constant coefficients, and that $\alpha^{m(p-s)}\beta^{ms}$, $s = 0, 1, \ldots, p$, are the distinct roots of the corresponding characteristic equation $g(y) = 0$, where

$$g(y) \equiv \prod_{s=0}^{p} (y - \alpha^{m(p-s)}\beta^{ms}),$$

which simplifies to (2.2) as follows:

Let $R_s = \alpha^{m(p-s)}\beta^{ms}$, $s = 0, 1, \ldots, p$. If $p = 1, 3, 5, \ldots$, there is an even number of roots, R_s, and thus $(p+1)/2$ pairs, $(y-R_j)\cdot(y-R_{p-j})$, $j = 0, 1, \ldots, (p-1)/2$. Since $\alpha\beta = c$, $V_n = \alpha^n + \beta^n$, $n = 0, 1, \ldots$, we have $R_j + R_{p-j} = c^{mj}V_{m(p-2j)}$ $\quad R_jR_{p-j} = c^{mp}$.

If $p = 2, 4, 6, \ldots$, there is an odd number of roots, R_s, and thus $p/2$ pairs, $(y-R)(y-R_{p-j})$, $j = 0, 1, \ldots, (p-2)/2$. The linear term, $y-R_{p/2} = y-c^{pm/2}$, accounts for the unpaired root, i.e., the middle root, $R_{p/2}$. This completes the proof of Lemma 1. Applications of (2.2) for $m = 1$ may be found in [1], [2], [3], and [4].

In terms of the translation operator, E, where $E^jQ_n = Q_{n+j}$, $j = 0, 1, \ldots$, set

$$u_n \equiv \left[\prod_{j=0}^{(p-2)/2} (E^2 - c^{mj}V_{m(p-2j)}E + c^{mp})\right] Q_n \quad (p = 2, 4, 6, \ldots).$$

Then, from (2.2), since $g(E)Q_n = (E-c^{pm/2})u_n = 0$, we have

$$u_n \equiv u_o c^{mpn/2} \quad (n = 0, 1, \ldots;\ p = 2, 4, \ldots). \tag{2.3}$$

We now define

$$\sum_{k=0}^{p} h_k^{(p)}(d/(2\sqrt{c}))y^{p-k} = \prod_{j=0}^{(p-2)/2} (y^2 - c^{mj}V_{m(p-2j)}y + c^{mp}) \quad (p = 2, 4, \ldots). \tag{2.4}$$

The coefficients $h_k^{(p)}(d/(2\sqrt{c}))$, $k = 0, 1, \ldots, p$, are also dependent on m, which is notationally suppressed for simplicity. Using (2.4), we may now rewrite (2.3) as

$$\sum_{k=0}^{p} h_k^{(p)}(d/(2\sqrt{c})) \prod_{s=1}^{p} W_{m(n+p-k)+i_s} \equiv c^{mpn/2} \sum_{k=0}^{p} h_k^{(p)}(d/(2\sqrt{c})) \prod_{s=1}^{p} W_{m(p-k)+i_s} \quad (n = 0, 1, \ldots;\ p = 2, 4, \ldots). \tag{2.5}$$

Let $p = 2q$, $q = 1, 2, \ldots$. Since $V_{2mk} = \alpha^{2mk} + \beta^{2mk}$ and $c = \alpha\beta$, we can write (2.4) as

$$\sum_{k=0}^{2q} h_{2q-k}^{(2q)}(d/(2\sqrt{c}))y^k = \prod_{k=1}^{q} (y^2 - c^{m(q-k)}V_{2mk}y + c^{2mq}) \tag{2.6}$$

$$= \prod_{k=1}^{q} (y - c^{m(q-k)}\alpha^{2mk})(y - c^{m(q-k)}\beta^{2mk})$$

$$= \prod_{k=1}^{q} \left[y - c^{mq}(\alpha/\beta)^{mk}\right] \left[y - c^{mq}(\alpha/\beta)^{-mk}\right] .$$

Set $y = c^{mq}x$ in (2.6), which now simplifies to

$$(2.7)\quad \sum_{k=0}^{2q} h_{2q-k}^{(2q)}(d/(2\sqrt{c}))c^{mqk}x^k = c^{2mq^2} \prod_{k=1}^{q} \left[x-(\alpha/\beta)^{mk}\right]\left[x-(\beta/\alpha)^{mk}\right]$$

We now define

$$(2.8)\quad b_k^{(2q)}(d/(2\sqrt{c})) \equiv c^{-mqk}h_k^{(2q)}(d/(2\sqrt{c})) \quad (k = 0, 1, \ldots, 2q) .$$

The, (2.7), with x replaced by y, now reads

$$(2.9)\quad \sum_{k=0}^{2q} b_k^{(2q)}(d/(2\sqrt{c}))y^{2q-k} \equiv \prod_{k=1}^{q} \left[y-(\alpha/\beta)^{mk}\right]\left[y-(\beta/\alpha)^{mk}\right]$$

$$= \prod_{k=1}^{q} (y^2 - c^{-mk}V_{2mk}y + 1)$$

$$(m, q = 1, 2, \ldots).$$

If we replace y by $(1/y)$ in (2.9), we conclude that

$$(2.10)\quad b_k^{(2q)}(d/(2\sqrt{c})) = b_{2q-k}^{(2q)}(d/(2\sqrt{c})) \qquad (k = 0, 1, \ldots, 2q).$$

Our results will be expressed in terms of $b_k^{(2q)}(d/(2\sqrt{c}))$. Recalling (1.2) and that $c = \alpha\beta$, we obtain from (2.9) for $y = 1$

$$(2.11)\quad \sum_{k=0}^{2q} b_k^{(2q)}(d/(2\sqrt{c})) = (-1)^q c^{-mq(q+1)/2} \prod_{k=1}^{q} (\alpha^{mk} - \beta^{mk})^2$$

$$= (-1)^q(\alpha - \beta)^{2q} c^{-mq(q+1)/2} \prod_{k=1}^{q} Z_{mk}^2$$

$$= (4c-d^2)^q c^{-mq(q+1)/2} \prod_{k=1}^{q} Z_{mk}^2 ,$$

since

$$(-1)^q(\alpha - \beta)^{2q} = \left[2\alpha\beta - (\alpha^2 - \beta^2)\right]^q = \left[2c - V_2\right]^q,$$

and

$$V_2 = dV_1 - cV_o = d^2 - 2c.$$

We will use (2.11) in the proof of Theorems 1 and 2.

3. TWO THEOREMS

Our first general result is as follows:

Theorem 1. Let W_o, W_1, $c \neq 0$, and $d \neq 0$ be arbitrary real numbers, and define W_n by (1.1). Let $n_o = 0, 1, \ldots; m, q = 1, 2, \ldots;$ and $r = 0, 1, \ldots, q$. Then, for $n = 0, 1, \ldots,$ we have

$$(3.1) \qquad c^{-mrn} \sum_{k=0}^{2q} c^{mrk}\, b_k^{(2q)}\,(d/(2\sqrt{c}))\, W^{2r}_{m(n+2q-k)+n_o}$$

$$= c^{rn_o + (mq(4r-q-1)/2}\binom{2r}{r}(4c - d^2)^{q-r}$$

$$\cdot\,(W_1^2 - dW_oW_1 + cW_o^2)^r \prod_{k=1}^{q} Z^2_{mk} \quad ,$$

where $b_k^{(2q)}(d/(2\sqrt{c}))$, $k = 0, 1, \ldots, 2q$, are defined by (2.9).

Proof. Since $\alpha \neq \beta$, the general solution to (1.1) is $W_n = A\alpha^n + B\beta^n$, $n = 0, 1, \ldots,$ where A and B are arbitrary constants whose values satisfy $W_o = A + B$ and $W_1 = A\alpha + B\beta$. We readily find that

$$(3.2) \qquad (\beta-\alpha)A = W_o\beta - W_1 \quad , \quad (\beta-\alpha)B = W_1 - \alpha W_o \quad .$$

Since $\alpha + \beta = d$, $c = \alpha\beta$, and $(\beta-\alpha)^2 = d^2 - 4c$, we obtain from (3.2)

$$(3.3) \qquad (d^2 - 4c)AB = -(W_1^2 - dW_oW_1 + cW_o^2)$$

Using the binomial theorem and then interchanging summations, we obtain

$$(3.4) \quad S \equiv c^{-mrn} \sum_{k=0}^{2q} c^{mr(2q-k)}\, b_{2q-k}^{(2q)}\,(d/(2\sqrt{c}))\, W^{2r}_{m(n+k)+n_o}$$

$$= c^{-mrn} \sum_{k=0}^{2q} (\alpha\beta)^{-mrk} b_{2q-k}^{(2q)}(d/(2\sqrt{c}))(A\alpha^{mn+mk+n_o} + B\beta^{mn+mk+n_o})^{2r}$$

$$= c^{mr(2q-n)} \sum_{s=0}^{2r} \binom{2r}{s} A^s B^{2r-s} (\alpha^s\beta^{2r-s})^{mn+n_o} G((\alpha/\beta)^{m(s-r)})$$

where, by (2.9) with $y = (\alpha/\beta)^{n(s-r)}$, we have

$$(3.5) \qquad G((\alpha/\beta)^{m(s-r)}) \equiv \sum_{k=0}^{2q} b_{2q-k}^{(2q)} (d/(2\sqrt{c})) \left[(\alpha/\beta)^{m(s-r)}\right]^k$$

$$= \prod_{k=1}^{q} \left[(\alpha/\beta)^{m(s-r)} - (\alpha/\beta)^{mk}\right] \cdot \left[(\alpha/\beta)^{m(s-r)} - (\alpha/\beta)^{-mk}\right] .$$

Since $0 \leq r \leq q$ and $0 \leq s \leq 2r$, we have $-q \leq s-r \leq q$. Thus, for $0 \leq s \leq 2r$, $s \neq r$, the sum in (3.5) vanishes; but for $s = r$, we obtain the non-zero term $G(1)$ (see (2.10), (2.11)). Thus, from (3.4), we obtain

$$(3.6) \qquad S = c^{2mrq+rn_o} \binom{2r}{r} (AB)^r \sum_{k=0}^{2q} b_k^{(2q)} (d/(2\sqrt{c})) ,$$

which yields the desired result with substitutions from (2.11) and (3.3)

The following general result yields Theorem 1 as an important special case:

<u>Theorem 2</u>. Let W_o, W_1, $c \neq 0$, and $d \neq 0$ be arbitrary real numbers and define W_n by (1.1). Let $m, q = 1, 2, \ldots,$ and $t_r = i_1 + i_2 + \ldots + i_{2r}$, where i_s, $s = 1, 2, \ldots, 2r$, $(r = 1, 2, \ldots, q)$, are positive integers or zero. Then, for $n = 0, 1, \ldots,$ we have

$$(3.7) \quad c^{-mrn} \sum_{k=0}^{2q} c^{mrk} b_k^{(2q)} (d/(2\sqrt{c})) \prod_{s=1}^{2r} W_{m(n+2q-k)+i_s}$$

$$= c^{mq(4r-q-1)/2} K_r (4c-d^2)^{q-r} (W_1^2 - dW_oW_1 + cW_o^2)^r \prod_{k=1}^{q} Z_{mk}^2 ,$$

with

$$(3.8)\qquad K_r = \sum_{j=1}^{\binom{2r-1}{r}} c^{\sigma(j,r)} V_{t_r - 2\sigma(j,r)} \qquad (r = 1, 2, \ldots, q) ,$$

$$(3.9)\quad \sigma(j,r) = i_1^{(j)} + i_2^{(j)} + i_3^{(j)} + \ldots + i_r^{(j)} \qquad (j = 1, 2, \ldots, \binom{2r-1}{r}) ,$$

where, for each j, $\sigma(j,r)$, as the sum of r integers, $i_s^{(j)}$, $s = 1, 2, \ldots, r$, represents one of the $\binom{2r-1}{r}$ combinations obtained by choosing r numbers from the $2r-1$ numbers, $i_1, i_2, i_3, \ldots, i_{2r-1}$.

<u>Proof.</u> From Lemma 1, we have

$$(3.10)\qquad Q_n = \prod_{s=1}^{2r} W_{mn+i_s} = \sum_{s=0}^{2r} C_s(\beta^{m(2r-s)}\alpha^{ms})^n ,$$

where C_s, $s = 0, 1, \ldots, 2r$, are arbitrary constants independent of n. Recalling the proof of Theorem 1, we have (see (3.7))

$$(3.11)\quad S \equiv c^{-mrn} \sum_{k=0}^{2q} c^{mr(2q-k)} b_{2q-k}^{(2q)}(d/(2\sqrt{c})) \sum_{s=0}^{2r} C_s(\beta^{m(2r-s)}\alpha^{ms})^{n+k}$$

$$= c^{-mrn+2mqr} \sum_{s=0}^{2r} C_s(\beta^{2r-s}\alpha^s)^{mn} \sum_{k=0}^{2q} b_{2q-k}^{(2q)}(d/(2\sqrt{c}))((\alpha/\beta)^{m(s-r)})^k$$

$$= c^{2mqr}\, C_r \sum_{k=0}^{2q} b_k^{(2q)}\,(d/(2\sqrt{c})) .$$

We proceed now to evaluate C_r. From (3.10), we have

$$(3.12)\qquad \prod_{s=1}^{2r} W_{mn+i_s} = \beta^{2mrn} \sum_{s=0}^{2r} C_s((\alpha/\beta)^{mn})^s ,$$

which is a polynomial in the variable $(\alpha/\beta)^{mn}$. Since $W_n = A\alpha^n + B\beta^n$, we have

$$W_{mn+i_s} = \left[A\alpha^{i_s}(\alpha/\beta)^{mn} + B\beta^{i_s}\right] ,$$

and thus

$$(3.13)\qquad \prod_{s=1}^{2r} W_{mn+i_s} = \beta^{2mrn} \prod_{s=1}^{2r} \left[A\alpha^{i_s}(\alpha/\beta)^{mn} + B\beta^{i_s}\right]$$

$$= \beta^{2mrn} A^{2r}\alpha^{t_r} \prod_{s=1}^{2r} \left[(\alpha/\beta)^{mn} + (B/A)(\beta/\alpha)^{i_s}\right] .$$

If we compare (3.12) and (3.13), and recall the definition of the elementary symmetric functions of the roots of a polynomial equation, we conclude that

$$(3.14)\qquad C_r = A^{2r}\alpha^{t_r}(-1)^r \sum_{k=1}^{\binom{2r}{r}} (-B/A)^r \prod_{s=1}^{r} (\beta/\alpha)^{i_{s,k}}$$

$$= (AB)^r \sum_{k=1}^{\binom{2r}{r}} \alpha^{\left(t_r - \sum_{s=1}^{r} i_{s,k}\right)} \beta^{\left(\sum_{s=1}^{r} i_{s,k}\right)} ,$$

where for each fixed k, $k = 1, 2, \ldots, \binom{2r}{r}$, each set of numbers, $i_{s,k}$, $s = 1, 2, \ldots, r$, is one of the $\binom{2r}{r}$ combinations obtained by choosing r numbers from the $2r$ numbers, i_s, $s = 1, 2, \ldots, 2r$. It should be noted that since (3.13) is a symmetric function in the variables i_s, $s = 1, 2, \ldots, 2r$, the role of i_{2r} in the definition of $\sigma(j, r)$ (see (3.9)) was a convenient choice. Since a choice of r numbers from a set of $2r$ numbers leaves another set of r numbers, we may pair off related terms of the sum in (3.14), noting our role assigned to i_{2r}. Thus, since $\binom{2r}{r} = 2\binom{2r-1}{r}$, and

$$\alpha^{t_r - \sigma(j,r)}\beta^{\sigma(j,r)} + \alpha^{\sigma(j,r)}\beta^{t_r - \sigma(j,r)} = c^{\sigma(j,r)} V_{t_r - 2\sigma(j,r)}$$

(see (1.3)), we have

$$(3.15)\qquad C_r = (AB)^r K_r \qquad (r = 1, 2, \ldots, q) .$$

Recalling definitions (2.11) and (3.3), we obtain our desired result (3.7) from (3.11).

Remarks. For $r = 2$, we have $\sigma(1,2) = i_1 + i_2$, $\sigma(2,2) = i_1 + i_3$, and $\sigma(3,2) = i_2 + i_3$.

For $r = 3$, we have

$$\begin{aligned}
\sigma(1,3) &= i_1 + i_2 + i_3, & \sigma(6,3) &= i_1 + i_4 + i_5,\\
\sigma(2,3) &= i_1 + i_2 + i_4, & \sigma(7,3) &= i_2 + i_3 + i_4,\\
\sigma(3,3) &= i_1 + i_2 + i_5, & \sigma(8,3) &= i_2 + i_3 + i_5,\\
\sigma(4,3) &= i_1 + i_3 + i_4, & \sigma(9,3) &= i_2 + i_4 + i_5,\\
\sigma(5,3) &= i_1 + i_3 + i_5, & \sigma(10,3) &= i_3 + i_4 + i_5,
\end{aligned}$$

If $i_s = n_o$, $s = 1, 2, \ldots, 2r$, then $t_r - 2\sigma(j,r) = 2rn_o - 2rn_o = 0$, $V_o = 2$, and $K_r = c^{rn_o}\binom{2r}{r}$. Thus, (3.7) yields (3.1) as a special case. Indeed, using the binomial theorem on $W_{mn+n_o} = A\alpha^{n_o}\alpha^{mn} + B\beta^{n_o}\beta^{mn}$, we obtain

$$W^{2r}_{mn+n_o} = \sum_{s=0}^{2r} \binom{2r}{s} A^s B^{2r-s} (\alpha^s\beta^{2r-s})^{n_o} (\beta^{m(2r-s)}\alpha^{ms})^n ,$$

where, (see (3.10)) $C_s = \binom{2r}{s} A^s B^{2r-s}(\alpha^s\beta^{2r-s})^{n_o}$, $s = 0, 1, \ldots, 2r$, and thus $C_r = c^{rn_o}\binom{2r}{r}(AB)^r$.

Consider the special case $i_s = n_o$, $s = 1, 2, \ldots, 2r-1$, and $i_{2r} \neq n_o$. Then $\sigma(j,r) \equiv rn_o$, $t_r = (2r-1)n_o + i_{2r}$, and thus (see (3.8))

$$K_r = c^{rn_o}\binom{2r-1}{r} V_{-n_o + i_{2r}} .$$

Next, consider the special case $i_s = n_o$, $s = 1, 2, \ldots, 2r-2$; $i_{2r-1} \neq i_{2r} \neq n_o$. Of the set of $\binom{2r-1}{r}$ combinations for $\sigma(j,r)$, there are $\binom{2r-2}{r-1}$ combinations which contain i_{2r-1}. For these cases, $\sigma(j,r) \equiv (r-1)n_o + i_{2r-1}$; and for the remaining $\binom{2r-1}{r} - \binom{2r-2}{r-1} = \binom{2r-2}{r}$

combinations, we have $\sigma(j, r) \equiv rn_o$. Thus, from (3.8), with $t_r = (2r-2)n_o + i_{2r-1} + i_{2r}$, we obtain

$$K_r = c^{(r-1)n_o + i_{2r-1}} \binom{2r-2}{r-1} V_{i_{2r} - i_{2r-1}} + c^{rn_o} \binom{2r-2}{r} V_{i_{2r-1} + i_{2r} - 2n_o} . \tag{3.16}$$

4. IDENTITIES FOR FIBONACCI SEQUENCES

Generalized Fibonacci numbers, H_n, are defined by $H_{n+2} = H_{n+1} + H_n$, $n = 0, 1, \ldots,$ where H_o and H_1 are arbitrary integers. In the notation of (1.2) and (1.3), we have $Z_n = F_n$, and $V_n = L_n$, the Lucas numbers. The following result is an application of Theorem 1, where $d = -c = 1$:

Theorem 3. Define (see (2.9))

$$\sum_{k=0}^{2q} b_k^{(2q)}(-i/2) y^{2q-k} = \prod_{k=1}^{q} (y^2 - (-1)^{mk} L_{2mk} y + 1) \quad (m, q = 1, 2, \ldots). \tag{4.1}$$

Let $n_o = 0, 1, \ldots;$ $m, q = 1, 2, \ldots;$ and $r = 0, 1, \ldots, q$. Then, for $n = 0, 1, \ldots,$ we have

$$(-1)^{mrn} \sum_{k=0}^{2q} (-1)^{mrk} b_k^{(2q)}(-i/2) H^{2r}_{m(n+2q-k)+n_o} = (-1)^{rn_o + (mq(q+1)/2)} \binom{2r}{r} (-5)^{q-r} (H_1^2 - H_oH_1 - H_o^2)^r \prod_{k=1}^{q} F^2_{mk} , \tag{4.2}$$

$$(-1)^{mrn} \sum_{k=0}^{2q} (-1)^{mrk} b_k^{(2q)}(-i/2) F^{2r}_{m(n+2q-k)+n_o} = (-1)^{rn_o + (mq(q+1)/2)} \binom{2r}{r} (-5)^{q} \prod_{k=1}^{q} F^2_{mk} , \tag{4.3}$$

$$(4.4) \qquad (-1)^{mrn} \sum_{k=0}^{2q} (-1)^{mrk} b_k^{(2q)} (-i/2) L^{2r}_{m(n+2q-k)+n_o}$$

$$= (-1)^{rn_o+(mq(q+1)/2)} \binom{2r}{r} (-5)^q \prod_{k=1}^{q} F^2_{mk} \quad .$$

Remarks. For the same values of r, n_o, m, and q, the constant term on the right-hand side of (4.4) is $(-5)^r$ times as great as the constant term on the right-hand side of (4.3)

In the examples given below, valid for $n = 0, 1, \ldots,$ we have set $D \equiv H_1^2 - H_oH_1 - H_o^2$. Applications of D in the ordering of Fibonacci sequences are given in [5] .

$$(4.5) \qquad (-1)^{mn} (H^2_{m(n+2)+n_o} - L_{2m}H^2_{m(n+1)+n_o} + H^2_{mn+n_o})$$

$$= 2(-1)^{m+n_o} DF_m^2 \qquad (n_o = 0, 1, \ldots; m = 1, 2, \ldots) \quad ,$$

$$(4.6) \qquad H^4_{n+4} - 4H^4_{n+3} - 19H^4_{n+2} - 4H^4_{n+1} + H^4_n = -6D^2 \quad ,$$

$$(4.7) \qquad (-1)^n(H^2_{n+4} + 4H^2_{n+3} - 19H^2_{n+2} + 4H^2_{n+1} + H^2_n) = 10D \quad ,$$

$$(4.8) \quad H_{n+4}H^3_{n+5} - 4H_{n+3}H^3_{n+4} - 19H_{n+2}H^3_{n+3} + 4H_{n+1}H^3_{n+2} + H_nH^3_{n+1}$$

$$= 3D^2 \quad ,$$

$$(4.9) \quad H^2_{n+4}H^2_{n+5} - 4H^2_{n+3}H^2_{n+4} - 19H^2_{n+2}H^2_{n+3} - 4H^2_{n+1}H^2_{n+2} + H^2_nH^2_{n+1} = D^2,$$

$$(4.10) \qquad (-1)^n(H^6_{n+6} - 14H^6_{n+5} - 90H^6_{n+4} + 350H^6_{n+3}$$

$$- 90H^6_{n+2} - 14H^6_{n+1} + H^6_n) = 80D^3 \quad ,$$

(4.11) $$H^4_{n+6} + 14H^4_{n+5} - 90H^4_{n+4} - 350H^4_{n+3} - 90H^4_{n+2} + 14H^4_{n+1} + H^4_n = -120D^2 ,$$

(4.12) $$(-1)^n(H^2_{n+6} - 14H^2_{n+5} - 90H^2_{n+4} + 350H^2_{n+3} - 90H^2_{n+2} - 14H^2_{n+1} + H^2_n) = 200D ,$$

(4.13) $$H^5_{n+6}H_{n+7} - 14H^5_{n+5}H_{n+6} - 90H^5_{n+4}H_{n+5} + 350H^5_{n+3}H_{n+4} - 90H^5_{n+2}H_{n+3} - 14H^5_{n+1}H_{n+2} + H^5_nH_{n+1} = 40(-1)^nD^3 ,$$

(4.14) $$H^3_{n+6}H^3_{n+7} - 14H^3_{n+5}H^3_{n+6} - 90H^3_{n+4}H^3_{n+5} + 350\,H^3_{n+3}H^3_{n+4} - 90\,H^3_{n+2}H^3_{n+3} - 14H^3_{n+1}H^3_{n+2} + H^3_nH^3_{n+1} = 20(-1)^{n+1}D^3 ,$$

(4.15) $$H^8_{n+8} - 33H^8_{n+7} - 747\,H^8_{n+6} + 3894\,H^8_{n+5} + 16270\,H^8_{n+4} + 3894\,H^8_{n+3} - 747\,H^8_{n+2} - 33H^8_{n+1} + H^8_n = 2520D^4 ,$$

(4.16) $$H^6_{n+8} + 33H^6_{n+7} - 747\,H^6_{n+6} - 3894\,H^6_{n+5} + 16270\,H^6_{n+4} - 3894\,H^6_{n+3} - 747\,H^6_{n+2} + 33H^6_{n+1} + H^6_n = 3600(-1)^{n+1}D^3 .$$

Two identities, (4.6) and a special case of (4.5), with $m = 1$ and $n_o = 0$, have been given previously in [6].

5. IDENTITIES FOR CHEBYSHEV POLYNOMIALS

Chebyshev polynomials [7, pp. 183-187] of the first kind, $T_n(x)$, and of the second kind, $U_n(x)$, are solutions of (1.1) when $d = 2x$ and $c = 1$. Thus, $W_n \equiv T_n(x)$ for $W_o = 1$, $W_1 = x$; $W_n \equiv U_n(x)$ for $W_o = 1$, $W_1 = 2x$; $Z_n \equiv U_{n-1}(x)$; and $V_n \equiv 2T_n(x)$.

We will now show that the Lucas functions Z_n and V_n of (1.1), where $c \neq 0$ and $d \neq 0$ are arbitrary real numbers, can be expressed in terms of Chebyshev polynomials as follows:

$$Z_{n+1} = c^{n/2} U_n(d/(2\sqrt{c})) \quad (n = 0, 1, \ldots), \tag{5.1}$$

$$V_n = 2c^{n/2} T_n(d/(2\sqrt{c})) \quad (n = 0, 1, \ldots). \tag{5.2}$$

Proof. Since $U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$, set $x = d/(2\sqrt{c})$ and then multiply both sides by $c^{(n+1)/2}$. Thus, using (5.1), we have $Z_o = 0$, $Z_1 = 1$, and $Z_{n+2} = dZ_{n+1} - cZ_n$, $n = 0, 1, \ldots$.

Since $T_{n+2}(x) = 2x\,T_{n+1}(x) - T_n(x)$, set $x = d/(2\sqrt{c})$ and then multiply both sides by $2c^{(n+2)/2}$. Thus, using (5.2), we have $V_o = 2$, $V_1 = d$, and $V_{n+2} = dV_{n+1} - cV_n$, $n = 0, 1, \ldots$.

The following result is an application of Theorem 1, where $d = 2x$ and $c = 1$:

Theorem 4. Define (see (2.9))

$$\sum_{k=0}^{2q} b_k^{(2q)}(x) y^{2q-k} = \prod_{k=1}^{q} (y^2 - 2T_{2mk}(x)y + 1) \quad (m, q = 1, 2, \ldots). \tag{5.3}$$

Let $n_o = 0, 1, \ldots$; $m, q = 1, 2, \ldots$; and $r = 0, 1, \ldots, q$. Then, for $n = 0, 1, \ldots$, we have

$$\sum_{k=0}^{2q} b_k^{(2q)}(x) T^{2r}_{m(n+2q-k)+n_o}(x) \tag{5.4}$$

$$= 4^{q-r} \binom{2r}{r} (1-x^2)^q \prod_{k=1}^{q} U^2_{mk-1}(x) \quad ,$$

$$(5.5)\quad \sum_{k=0}^{2q} b_k^{(2q)}(x)U^{2r}_{m(n+2q-k)+n_o}(x) = 4^{q-r}\binom{2r}{r}(1-x^2)^{q-r}\prod_{k=1}^{q} U^2_{mk-1}(x) .$$

Remarks. Identities (5.4) and (5.5) yield trigonometric identities by recalling that if $x = \cos\theta$, then $T_n(\cos\theta) = \cos(n\theta)$ and $U_n(\cos\theta) = \sin(n+1)\theta/(\sin\theta)$. Since $\sin(i\theta) = i\sinh\theta$ and $\cos(i\theta) = \cosh\theta$, identities for the hyperbolic functions are then obtained from the corresponding trigonometric identities. Additional complicated identities can be obtained from (5.4) and (5.5) by differentiation with respect to x. Some sample identities, valid for $n = 0, 1, \ldots,$ are given below:

$$(5.6)\quad T^2_{m(n+2)+n_o}(x) - 2T_{2m}(x)T^2_{m(n+1)+n_o}(x) + T^2_{mn+n_o}(x)$$
$$= 2(1-x^2)U^2_{m-1}(x) \quad (m = 1, 2, \ldots;\ n_o = 0, 1, \ldots),$$

$$(5.7)\quad T^4_{n+4}(x) - (16x^4-12x^2)T^4_{n+3}(x) + (64x^6-96x^4+40x^2-2)T^4_{n+2}(x)$$
$$-(16x^4-12x^2)T^4_{n+1}(x) + T^4_n(x) = 24x^2(1-x^2)^2 ,$$

$$(5.8)\quad T^3_{n+4}(x)T_{n+5}(x) - (16x^4-12x^2)T^3_{n+3}(x)T_{n+4}(x)$$
$$+(64x^6-96x^4+40x^2-2)T^3_{n+2}(x)T_{n+3}(x) - (16x^4-12x^2)T^3_{n+1}(x)T_{n+2}(x)$$
$$+ T^3_n(x)T_{n+1}(x) = 24x^3(1-x^2)^2 .$$

Let

$$A_1(x) = 64x^6 - 80x^4 + 24x^2 - 2 ,$$

$$A_2(x) = 1024x^{10} - 2304x^8 + 1792x^6 - 560x^4 + 64x^2 - 1 ,$$

$$A_3(x) = 4096x^{12} - 12288x^{10} + 14080x^8 - 7552x^6 + 1856x^4 - 176x^2 + 4 .$$

Then

$$(5.9)\quad T^6_{n+6}(x) - A_1(x)T^6_{n+5}(x) + A_2(x)T^6_{n+4}(x) - A_3(x)T^6_{n+3}(x)$$
$$+ A_2(x)T^6_{n+2}(x) - A_1(x)T^6_{n+1}(x) + T^6_n(x) = 80x^2(1-x^2)^3(4x^2-1)^2 ,$$

(5.10) $$T^4_{n+6}(x) - A_1(x)T^4_{n+5}(x) + A_2(x)T^4_{n+4}(x) - A_3(x)T^4_{n+3}(x)$$

$$+ A_2(x)T^4_{n+2}(x) - A_1(x)T^4_{n+1}(x) + T^4_n(x) = 96x^2(1-x^2)^3(4x^2-1)^2 ,$$

(5.11) $$T^3_{n+6}(x)T^3_{n+7}(x) - A_1(x)T^3_{n+5}(x)T^3_{n+6}(x) + A_2(x)T^3_{n+4}(x)T^3_{n+5}(x)$$

$$- A_3(x)T^3_{n+3}(x)T^3_{n+4}(x) + A_2(x)T^3_{n+2}(x)T^3_{n+3}(x) - A_1(x)T^3_{n+1}(x)T^3_{n+2}(x)$$

$$+ T^3_n(x)T^3_{n+1}(x) = 16x^3(2x^2+3)(1-x^2)^3(4x^2-1)^2 .$$

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OMISSION AND INFORMATION

The "Factorization of 36 Fibonacci Numbers F_n with $n > 100$" by L. A. G. Dresel and D. E. Daykin should have included the following references.

1. Dov Jarden Recurring Sequences, Israel, 1958, contains many factorizations of first 385 L_n and F_n. This is being reissued soon and will be available again from the Fibonacci Association.

2. Brother U. Alfred and John Brillhart "Fibonacci Century Mark Reached" FQJ, Vol. I, No. 1, p. 45, Feb., 1963.

3. Brother U. Alfred "Fibonacci Discovery" contains factors of first 100 F_n and first 50 L_n. See ad this issue page 291.

The factors available now allows one to factor higher Fibonacci Numbers since $F_{2n} = L_n F_n$.

John Brillhart reports that in a short time he will have published a report containing all the prime factors less than 2^{30} of F_n for $n < 2000$ and of L_n for $n < 1000$. This is exciting news.