ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-462 *Proposed by Ioan Sadoveanuv, Ellensburg, WA*

Let $G(x) = x^k + a_1x^{k-1} + \cdots + a_k$ be a polynomial with c a root of order p. If $G^{(p)}(x)$ denotes the p^{th} derivative of $G(x)$, show that $\{n^pc^{n-p}/G^{(p)}(c)\}$ is a solution of the recurrence $u_n = c^{n-k} - a_1u_{n-1} - a_2u_{n-2} - \cdots - a_ku_{n-k}$.

H-463 *Proposed by Paul Bruckman, Edmonds, WA*

Establish the identity

$$(1)\qquad \sum_{n=1}^{\infty}\Phi(n)\left(\frac{z^n}{1-z^{2n}}\right) = \frac{z(1+z+z^2)}{(1-z^2)^2},\quad z \in \mathbb{C},\ |z| < 1,$$

and Φ is the Euler (totient) function.

As special cases of (1), obtain the following identities:

$$(2)\qquad \sum_{n=1}^{\infty}\Phi(2n)/F_{2ns} = \sqrt{5}/L_s^2,\quad s = 1, 3, 5, \ldots;$$

$$(3)\qquad \sum_{n=1}^{\infty}\Phi(2n-1)/L_{(2n-1)s} = F_s\sqrt{5}/L_s^2,\quad s = 1, 3, 5, \ldots;$$

$$(4)\qquad \sum_{n=1}^{\infty}\Phi(n)/F_{ns} = (L_s+1)/F_s^2\sqrt{5},\quad s = 2, 4, 6, \ldots;$$

$$(5)\qquad \sum_{n=1}^{\infty}(-1)^{n-1}\Phi(n)/F_{ns} = (L_s-1)/F_s^2\sqrt{5},\quad s = 2, 4, 6, \ldots;$$

$$(6)\qquad \sum_{n=1}^{\infty}(-1)^{n-1}\Phi(2n)/F_{2ns} = \begin{cases}1/F_s^2\sqrt{5}, & s = 1, 3, 5, \ldots;\\ \sqrt{5}/L_s^2, & s = 2, 4, 6, \ldots;\end{cases}$$

$$(7)\qquad \sum_{n=1}^{\infty}(-1)^{n-1}\Phi(2n-1)/F_{(2n-1)s} = L_s/F_s^2\sqrt{5},\quad s = 1, 3, 5, \ldots;$$

$$(8)\qquad \sum_{n=1}^{\infty}(-1)^{n-1}\Phi(2n-1)/L_{(2n-1)s} = F_s\sqrt{5}/L_s^2,\quad s = 2, 4, 6, \ldots .$$

H-464 *Proposed by H.-J. Seiffert, Berlin, Germany*

Show that

$$\sum_{k=0}^{[n/2]} \binom{n}{k} A_{n-2k} = F_n,$$

where

$$A_j = (-1)^{[(j+2)/5]} - \left((-1)^{[j/5]} + (-1)^{[(j+4)/5]}\right)/2.$$

[ ] denotes the greatest integer function.

H-465 *Proposed by Richard André-Jeannin, Tunisia*

Let p be a prime number, and let $r_1, r_2, \ldots, r_s$ be natural integers such that $s \geq 2$, $r_1 < p$, and

$$\sum_{k=1}^{k=s} kr_k = p.$$

Show that the number

$$B_{r_1, r_2, \ldots, r_s} := \frac{1}{r_1 + r_2 + \cdots + r_s} \frac{(r_1 + r_2 + \cdots + r_s)!}{r_1! r_2! \cdots r_s!}$$

is an integer.

SOLUTIONS

An Odd Problem

H-442 *Proposed by Piero Filipponi, Rome, Italy (Vol. 28, no. 2, May 1990)*

Prove that the congruence

$$\prod_{i=1}^{(d-3)/2} (2i+1)^2 \equiv \begin{cases} 1 \pmod{d} & \text{if } (d+1)/2 \text{ is even} \\ -1 \pmod{d} & \text{if } (d+1)/2 \text{ is odd} \end{cases}$$

holds if and only if d is an odd prime.

Solution by the proposer

Let n be an even integer. The equality

$$(1) \qquad n! = \frac{1}{2^n} \prod_{i=0}^{(n-2)/2} [n(n+2) - 4i(i+1)]$$

can be proved readily by writing $n = 2m$ and rewriting (1) as

$$(2m)! = \frac{1}{2^{2m}} \prod_{i=0}^{m-1} [4m(m+1) - 4i(i+1)] = \prod_{i=0}^{m-1} (m-i)(m+i+1).$$

Let d be an odd integer. By (1) we have

$$(2) \qquad (d-1)! = \frac{1}{2^{d-1}} \prod_{i=0}^{(d-3)/2} [d^2 - 1 - 4i(i+1)].$$

If d is a prime, by using Fermat's little theorem, we obtain the congruence

$$(3) \qquad (d-1)! \equiv \prod_{i=0}^{(d-3)/2} (-1-4i^2-4i) = \prod_{i=0}^{(d-3)/2} [-(2i+1)^2] \pmod{d}.$$

By using Wilson's theorem, ($(d-1)! \equiv -1 \pmod{d}$ iff d is prime); thus, by (3) we get

$$\prod_{i=0}^{(d-3)/2} [-(2i+1)^2] = (-1)^{(d-1)/2} \prod_{i=1}^{(d-3)/2} (2i+1)^2 \equiv -1 \pmod{d}$$

iff d is prime, that is,

$$\prod_{i=1}^{(d-3)/2} (2i+1)^2 \equiv (-1)^{(d+1)/2} \pmod{d} \text{ iff } d \text{ is prime.}$$

Also solved by P. Bruckman, R. Hendel, and L. Somer.

Another Odd One

H-443 *Proposed by Richard André-Jeannin, Tunisia*
(Vol. 28, no. 3, August 1990)

Let us consider the recurrence

$$w_n = mw_{n-1} + w_{n-2},$$

where $m > 0$ is an integer and U_n, V_n the solutions defined by

$$U_0 = 0,\ U_1 = 1;\ V_0 = 2,\ V_1 = m.$$

Show that, if q is an odd divisor of $m^2 + 1$, then

$$V_q \equiv m \pmod{q}.$$

Solution by H.-J. Seiffert, Berlin, Germany

First, we prove that

$$(1) \qquad V_q \equiv m^k V_{q-3k} \pmod{q},\ k = 0, \ldots, [q/3],$$

where [ ] denotes the greatest integer function.

Obviously, (1) is true for $k = 0$. Assuming that it holds for k, where

$$0 \le k < [q/3],$$

we obtain

$$\begin{aligned} V_q \equiv m^k V_{q-3k} &= m^k(mV_{q-3k-1} + V_{q-3k-2}) \\ &= m^k(m^2+1)V_{q-3k-2} + m^{k+1}V_{q-3k-3} \\ &\equiv m^{k+1}V_{q-3(k+1)} \pmod{q}. \end{aligned}$$

This completes the induction proof of (1).

For any odd prime divisor p of q, the congruence $m^2 \equiv -1 \pmod{p}$ shows that -1 is a quadratic residue mod p; hence (see T. M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, 1976, Theorem 9.4, p. 181), $p \equiv 1 \pmod{4}$. This holds for any odd prime divisor p of q. Since q is odd, we also have $q \equiv 1 \pmod{4}$. In (1), we take $k = [q/3]$. Hence, we have $0 \le q - 3k \le 2$. The case $q = 3k$ would imply $m^2 \equiv -1 \pmod{3}$, which contradicts Fermat's little theorem. If $q = 3k + 1$, then $q \equiv 1 \pmod{4}$ implies that k is divisible by 4. From $m^2 = -1 \pmod{q}$ and (1), we get

$$V_q \equiv m^k V_1 = m^{k+1} \equiv (-1)^{k/2} m = m \pmod{q}.$$

If $q = 3k + 2$, then q odd and $q \equiv 1 \pmod 4$ yield $k \equiv 1 \pmod 4$. Now $m^2 \equiv -1 \pmod q$ and (1) give

$$V_q \equiv m^k V_2 = m^k(m^2 + 2) = m^k(m^2 + 1) + m^k \equiv m^k$$
$$\equiv (-1)^{(k-1)/2} m = m \pmod q.$$

This completes the solution. Finally, it should be noted that (1) also follows from the identity

$$V_q = (m^2 + 1) \sum_{j=0}^{k-1} m^j V_{q-3j-2} + m^k V_{q-3k},$$

valid for $k = 0, \ldots, [q/3]$.

Also solved by P. Bruckman, F. Howard, L. Somer, and the proposer.

Summing It Up

H-444 *Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 28, no. 3, August 1990)*

Show that, for $n = 0, 1, 2, \ldots,$

$$F_n = \sum_{\substack{k=0 \\ (5,\, n-2k)=1}}^{[n/2]} (-1)^{[(n-2k+2)/5]} \binom{n}{k},$$

where (r, s) denotes the greatest common divisor of r and s and $[\ ]$ the greatest integer function.

Solution by Paul Bruckman, Edmonds, WA

We employ a generating function technique to prove what appears to be a very remarkable identity. Define

$$\text{(1)} \qquad G_n = \sum_{\substack{k=0 \\ (5,\, n-2k)=1}}^{\left[\frac{1}{2}n\right]} (-1)^{\left[\frac{1}{5}(n-2k+2)\right]} \binom{n}{k}, \quad n = 0, 1, 2, \ldots,$$

and

$$\text{(2)} \qquad g(x) = \sum_{n=0}^{\infty} G_n x^n.$$

Then (formally, at least),

$$g(x) = \sum_{\substack{n,\, k=0 \\ (n,5)=1}}^{\infty} (-1)^{\left[\frac{1}{5}(n+2)\right]} x^{n+2k} \binom{n+2k}{k}.$$

Now

$$\sum_{k=0}^{\infty} \binom{n+2k}{k} x^{2k} = \sum_{k=0}^{\infty} \binom{n+2k}{2k}\binom{2k}{k} \Big/ \binom{n+k}{k} x^{2k} = \sum_{k=0}^{\infty} \frac{(n+1)_{2k}}{(n+1)_k} \cdot \frac{x^{2k}}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}(n+1)\right)_k \left(\frac{1}{2}(n+2)\right)_k}{(n+1)_k} \cdot \frac{(2x)^{2k}}{k!}$$

$$= {}_2F_1\left[\begin{matrix} \frac{n+1}{2}, \frac{n+2}{2} ; 4x^2 \\ n+1 \end{matrix}\right]$$

(the "standard" hypergeometric function). Hence,

$$(3) \qquad g(x) = \sum_{\substack{n=0 \\ (n,5)=1}}^{\infty} (-1)^{\left[\frac{1}{5}(n+2)\right]} x^n \cdot {}_2F_1\left[\begin{matrix} \frac{n+1}{2}, \frac{n+2}{2} \\ n+1 \end{matrix}; 4x^2\right].$$

We will use the following known transformations of the hypergeometric function:

$$(4) \qquad {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; w\right] = (1-w)^{-a} {}_2F_1\left[\begin{matrix} a, c-b \\ c \end{matrix}; \frac{-w}{1-w}\right],$$

(Theorem 20, p. 60, of [1])

$$(5) \qquad {}_2F_1\left[\begin{matrix} a, b \\ a+b+\frac{1}{2} \end{matrix}; 4z(1-z)\right] = {}_2F_1\left[\begin{matrix} 2a, 2b \\ a+b+\frac{1}{2} \end{matrix}; z\right].$$

(Theorem 25, p. 67, of [1])

In (4), let

$$a = \frac{1}{2}(n+1), \; b = \frac{1}{2}(n+2), \; c = n+1, \; w = 4x^2.$$

We thus obtain

$$(6) \qquad {}_2F_1\left[\begin{matrix} \frac{n+1}{2}, \frac{n+2}{2} \\ n+1 \end{matrix}; 4x^2\right] = (1-4x^2)^{-\frac{1}{2}(n+1)} \cdot {}_2F_1\left[\begin{matrix} \frac{n+1}{2}, \frac{n}{2} \\ n+1 \end{matrix}; \frac{-4x^2}{1-4x^2}\right].$$

In (5), let

$$a = \frac{1}{2}(n+1), \; b = \frac{1}{2}n, \; z = \frac{1}{2}(1-\theta),$$

where $\theta = \theta(x) = (1-4x^2)^{-\frac{1}{2}}$; note that

$$4z(1-z) = (1-\theta)(1+\theta) = \frac{-4x^2}{1-4x^2}.$$

Then

$${}_2F_1\left[\begin{matrix} \frac{n+1}{2}, \frac{n}{2} \\ n+1 \end{matrix}; \frac{-4x^2}{1-4x^2}\right] = {}_2F_1\left[\begin{matrix} n+1, n \\ n+1 \end{matrix}; z\right] = {}_1F_0\left[\begin{matrix} n \\ - \end{matrix}; z\right]$$

$$= \sum_{k=0}^{\infty} \frac{(n)_k}{k!} z^k = \sum_{k=0}^{\infty} \binom{-n}{k}(-z)^k = (1-z)^{-n}.$$

Therefore, using (3) and (6):

$$g(x) = \sum_{\substack{n=0 \\ (n,5)=1}}^{\infty} (-1)^{\left[\frac{1}{5}(n+2)\right]} x^n (1-z)^{-n} \theta^{n+1}.$$

We now make the substitution, $y = \theta x(1-z)^{-1}$. Thus,

$$y = 2x\theta(1+\theta)^{-1} = 2x\theta(1-\theta)(1-\theta^2)^{-1} = \frac{-2x(1-\theta^{-1})}{-4x^2},$$

or

(7) $$y = \frac{1 - (1 - 4x^2)^{\frac{1}{2}}}{2x}.$$

We then obtain

(8) $$g(x) = \theta \sum_{\substack{n=0 \\ (n,5)=1}}^{\infty} (-1)^{\left[\frac{1}{5}(n+2)\right]} y^n, \text{ where } y \text{ is given by (7).}$$

Next, we obtain a closed form for $g(x)$, as follows:

$$g(x) = \theta \sum_{m=0}^{\infty} \sum_{r=1}^{4} (-1)^{\left[\frac{1}{5}(5m+r+2)\right]} y^{5m+r} = \theta \sum_{m=0}^{\infty} (-1)^m y^{5m} \sum_{r=1}^{4} (-1)^{\left[\frac{1}{5}(r+2)\right]} y^r$$

$$= \theta \sum_{m=0}^{\infty} (-y^5)^m (y + y^2 - y^3 - y^4) = \frac{\theta y(1 + y)(1 - y^2)}{1 + y^5};$$

hence,

(9) $$g(x) = \theta y(1 - y^2)(1 - y + y^2 - y^3 + y^4)^{-1}.$$

To evaluate $g(x)$ as an explicit function of x, we employ the readily verifiable result:

(10) $$y^2 = y/x - 1.$$

Using (10), we obtain

$$y^3 = y^2/x - y = -y + \frac{1}{x}(y/x - 1) = \frac{-1}{x} + (1/x^2 - 1)y;$$

$$y^4 = -y/x + (1/x^2 - 1)y^2 = -y/x + (1/x^2 - 1)(y/x - 1)$$

$$= 1 - x^{-2} + \frac{y}{x}(x^{-2} - 2).$$

Therefore,

$$(1 - y + y^2 - y^3 + y^4) = 1 + x^{-1} - x^{-2} - y(x^{-1} + x^{-2} - x^{-3}),$$

after simplification, or

(11) $$1 - y + y^2 - y^3 + y^4 = x^{-3}(y - x)(1 - x - x^2).$$

Also,

$$\theta y(1 - y^2) = \theta y + \theta(x^{-1} + (1 - x^{-2})y) = -\theta(y - x)x^{-2} + 2\theta y.$$

Therefore,

(12) $$g(x) = x\theta\left(\frac{2x^2 y}{y - x} - 1\right)(1 - x - x^2)^{-1}.$$

From (10), $xy^2 = y - x$; therefore,

$$\frac{2x^2y}{y - x} - 1 = \frac{2x^2y}{xy^2} - 1 = \frac{2x - y}{y} = \frac{4x^2 - 1 + (1 - 4x^2)^{\frac{1}{2}}}{1 - (1 - 4x^2)^{\frac{1}{2}}} = (1 - 4x^2)^{\frac{1}{2}} = \theta^{-1}.$$

Hence, we finally obtain

(13) $$g(x) = x(1 - x - x^2)^{-1}.$$

We recognize $g(x)$ in (13) as the generating function of the Fibonacci numbers; more specifically,

(14) $$g(x) = \sum_{n=0}^{\infty} F_n x^n.$$

Comparison with (2) yields the desired result:

(15) $G_n = F_n$, $n = 0, 1, 2, \ldots$. Q.E.D.

Reference:

1. E. D. Rainville. *Special Functions*. New York: Chelsea, 1960.

Also solved by S. Rizavi and the proposer.

Mu-ve Over

H-445 *Proposed by Paul S. Bruckman, Edmonds, WA (Vol. 28, no. 3, August 1990)*

Please refer to the volume of *The Fibonacci Quarterly* cited above for a complete statement of this problem.

Solution by Y. H. Harris Kwong, SUNY College at Fredonia, Fredonia, NY

For $|z| < 1$, we have

$$(*) \qquad \sum_{n=1}^{\infty} \mu(n)\left(\frac{z^n}{1 - z^{2n}}\right) = \sum_{n=1}^{\infty} \mu(n) \sum_{k=1}^{\infty} z^{(2k-1)n} = \sum_{n=1}^{\infty}\left(\sum_{\substack{d|n \\ d \text{ odd}}} \mu(n/d)\right) z^n.$$

Let $n = 2^s t$, where t is odd and $s \geq 0$. Then the set of odd divisors of n is precisely the set of divisors of t. Thus,

$$\sum_{\substack{d|n \\ d \text{ odd}}} \mu(n/d) = \mu(2^s)\sum_{d|t} \mu(t/d) = \begin{cases} 1 & \text{if } s = 0 \text{ and } t = 1; \\ -1 & \text{if } s = t = 1; \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, $(2k - 1)n$ in $(*)$ is odd iff n is odd. Hence, we conclude that for $|z| < 1$,

$$\sum_{n \text{ odd}} \mu(n)\left(\frac{z^n}{1 - z^{2n}}\right) = z \quad \text{and} \quad \sum_{n \text{ even}} \mu(n)\left(\frac{z^n}{1 - z^{2n}}\right) = -z^2,$$

which lead to (1). For (2)-(7), we shall use the identities:

$$\frac{1}{\sqrt{5}}\frac{1}{F_{ms}} = \frac{\beta^{ms}}{(-1)^{ms} - \beta^{2ms}} \quad \text{and} \quad \frac{1}{L_{ms}} = \frac{\beta^{ms}}{(-1)^{ms} + \beta^{2ms}}.$$

$$(2) \quad \text{For } s = 1,2,3,\ldots,\ \frac{1}{\sqrt{5}}\sum_{n=1}^{\infty} \frac{\mu(2n)}{F_{2ns}} = \sum_{m \text{ even}} \mu(m)\left(\frac{\beta^{ms}}{1 - \beta^{2ms}}\right) = -\beta^{2s}.$$

$$(3) \quad \text{For } s = 1,3,5,\ldots,\ \sum_{n=1}^{\infty} \frac{\mu(2n-1)}{L_{(2n-1)s}} = -\sum_{m \text{ odd}} \mu(m)\left(\frac{\beta^{ms}}{1 - \beta^{2ms}}\right) = -\beta^{s}.$$

$$(4) \quad \text{For } s = 2,4,6,\ldots,\ \frac{1}{\sqrt{5}}\sum_{n=1}^{\infty} \frac{\mu(n)}{F_{ns}} = \sum_{n=1}^{\infty} \mu(n)\left(\frac{\beta^{ns}}{1 - \beta^{2ns}}\right) = \beta^{s} - \beta^{2s}.$$

$$(5) \quad \text{For } s = 2,4,6,\ldots,\ \frac{1}{\sqrt{5}}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\mu(n)}{F_{ns}} = -\sum_{n=1}^{\infty} \mu(n)\left(\frac{(-\beta^s)^n}{1 - (-\beta^s)^{2n}}\right) = \beta^{s} + \beta^{2s}.$$

$$(6) \quad \text{For } s = 1,3,5,\ldots,\ \frac{1}{\sqrt{5}}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\mu(2n-1)}{F_{(2n-1)s}} = -\frac{1}{i}\sum_{m \text{ odd}} \mu(m)\left(\frac{(i\beta^s)^m}{1 - (i\beta^s)^{2m}}\right) = -\beta^{s}.$$

$$(7) \quad \text{For } s = 2,4,6,\ldots,\ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\mu(2n-1)}{L_{(2n-1)s}} = \frac{1}{i}\sum_{m \text{ odd}} \mu(m)\left(\frac{(i\beta^s)^m}{1 - (i\beta^s)^{2m}}\right) = \beta^{s}.$$

Also solved by C. Georghiou, H.-J. Seiffert, and the proposer.

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