RECURRENT FORMULAS OF THE GENERALIZED FIBONACCI AND TRIBONACCI SEQUENCES

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In [1], it was shown that there are 36 different schemes of generalization for the Fibonacci sequence in the case of three sequences. Ten of these are trivial and the other 26 are grouped into seven classes. The elements of each class are equivalent with exactness up to a substitution of their members. Thus, for every class, we shall give the recurrent formulas of the members of one of its schemes, using the notation in [1].

Everywhere, let

$$a_0 = C_1,\ b_0 = C_2,\ c_0 = C_3,\ a_1 = C_4,\ b_2 = C_5,\ c_2 = C_6,$$

and assume that $n \geq 0$ is a natural number where $C_1, C_2, \ldots, C_6$ are given constants and x is one of the symbols a, b, and c. Class I contains the schemes S_6 and S_9, where

$$S_6\colon \begin{cases} a_{n+1} = a_{n+2} + b_n \\ b_{n+2} = b_{n+1} + c_n \\ c_{n+2} = c_{n+1} + a_n. \end{cases}$$

The recurrent formula for this scheme is

$$x_{n+6} = 3x_{n+5} - 3x_{n+4} + x_{n+3} + x_n,$$

that is,

$$\begin{aligned} a_{n+6} &= 3a_{n+5} - 3a_{n+4} + a_{n+3} + a_n, \\ b_{n+6} &= 3b_{n+5} - 3b_{n+4} + b_{n+3} + b_n, \\ c_{n+6} &= 3c_{n+5} - 3c_{n+4} + c_{n+3} + c_n. \end{aligned}$$

Class II contains the schemes S_{15} and S_{25}, where

$$S_{15}\colon \begin{cases} a_{n+2} = b_{n+1} + a_n \\ b_{n+2} = c_{n+1} + b_n \\ c_{n+2} = a_{n+1} + c_n. \end{cases}$$

The recurrent formula for this scheme is

$$x_{n+5} = 3x_{n+4} + x_{n+3} - 3x_{n+2} + x_n.$$

Class III contains the schemes S_{20} and S_{33}, where

$$S_{20}\colon \begin{cases} a_{n+2} = b_{n+1} + b_n \\ b_{n+2} = c_{n+1} + c_n \\ c_{n+2} = a_{n+1} + a_n. \end{cases}$$

The recurrent formula for this scheme is

$$x_{n+6} = x_{n+3} + 3x_{n+2} + 3x_{n+1} + x_n.$$

Class IV contains the schemes S_{23} and S_{33}, where

$$S_{23}: \begin{cases} a_{n+2} = b_{n+1} + c_n \\ b_{n+2} = c_{n+1} + a_n \\ c_{n+2} = a_{n+1} + b_n. \end{cases} \quad \text{(cf. [3])}$$

The recurrent formula for this scheme is

$$x_{n+6} = 4x_{n+3} + x_n.$$

Class V contains the schemes S_7, S_{12}, S_{14}, S_{22}, S_{28}, and S_{31}, where

$$S_7: \begin{cases} a_{n+2} = a_{n+1} + b_n \\ b_{n+2} = c_{n+1} + a_n \\ c_{n+2} = b_{n+1} + c_n. \end{cases}$$

The recurrent formula for this scheme is

$$x_{n+6} = x_{n+5} + 2x_{n+4} - 2x_{n+3} + x_{n+2} - x_n.$$

Class VI contains the schemes S_8, S_{11}, S_{18}, S_{21}, S_{32}, and S_{35}, where

$$S_8: \begin{cases} a_{n+2} = a_{n+1} + b_n \\ b_{n+2} = c_{n+1} + c_n \\ c_{n+2} = b_{n+1} + a_n. \end{cases}$$

The recurrent formula for this scheme is

$$x_{n+6} = x_{n+5} + x_{n+4} = x_{n+2} + x_{n+1} + x_n.$$

Class VII contains the schemes S_{16}, S_{19}, S_{24}, S_{26}, S_{29}, and S_{34}, where

$$S_{16}: \begin{cases} a_{n+2} = b_{n+1} + a_n \\ b_{n+2} = c_{n+1} + c_n \\ c_{n+2} = a_{n+1} + b_n. \end{cases}$$

The recurrent formula for this scheme is

$$x_{n+6} = x_{n+4} + 2x_{n+3} + 2x_{n+2} - x_{n+1} - x_n.$$

Using the data given above and some ideas from [3], we can construct eight different schemes of generalized Tribonacci sequences in the case of two sequences. We introduce their recurrent formulas below.

Everywhere let

$$a_0 = C_1,\ b_0 = C_2,\ a_1 = C_3,\ b_1 = C_4,\ a_2 = C_5,\ b_2 = C_6,$$

and assume that $n \geq 0$ is a natural number, where $C_1, C_2, \ldots, C_6$ are given constants and x is one of the symbols a or b.

The different schemes are as follows:

$$T_1: \begin{cases} a_{n+3} = a_{n+2} + a_{n+1} + a_n \\ b_{n+3} = b_{n+2} + b_{n+1} + b_n \end{cases}, \qquad T_2: \begin{cases} a_{n+3} = a_{n+2} + a_{n+1} + b_n \\ b_{n+3} = b_{n+2} + b_{n+1} + a_n \end{cases},$$

$$T_3: \begin{cases} a_{n+3} = a_{n+2} + b_{n+1} + a_n \\ b_{n+3} = b_{n+2} + a_{n+1} + b_n \end{cases}, \qquad T_4: \begin{cases} a_{n+3} = a_{n+2} + b_{n+1} + b_n \\ b_{n+3} = b_{n+2} + a_{n+1} + a_n \end{cases},$$

$$T_5: \begin{cases} a_{n+3} = b_{n+2} + a_{n+1} + a_n \\ b_{n+3} = a_{n+2} + b_{n+1} + b_n \end{cases}, \qquad T_6: \begin{cases} a_{n+3} = b_{n+2} + a_{n+1} + b_n \\ b_{n+3} = a_{n+2} + b_{n+1} + a_n \end{cases},$$

$$T_7: \begin{cases} a_{n+3} = b_{n+2} + b_{n+1} + a_n \\ b_{n+3} = a_{n+2} + a_{n+1} + b_n \end{cases}, \qquad T_8: \begin{cases} a_{n+3} = b_{n+2} + b_{n+1} + b_n \\ b_{n+3} = a_{n+2} + a_{n+1} + a_n \end{cases} \text{(cf. [3])}.$$

The first scheme is trivial. All of the others are nontrivial; they have the following recurrent formulas for $n \geq 0$:

- for T_2: $x_{n+6} = 2x_{n+5} + x_{n+4} - 2x_{n+3} - x_{n+2} + x_n$,
- for T_3: $x_{n+6} = 2x_{n+5} - x_{n+4} + 2x_{n+3} - x_{n+2} - x_n$;
- for T_4: $x_{n+6} = 2x_{n+5} - x_{n+4} + x_{n+2} + 2x_{n+1} + x_n$;
- for T_5: $x_{n+6} = 3x_{n+4} + 2x_{n+3} - x_{n+2} - 2x_{n+1} - x_n$;
- for T_6: $x_{n+6} = 3x_{n+4} + x_{n+2} + x_n$;
- for T_7: $x_{n+6} = x_{n+4} + 4x_{n+3} + x_{n+2} - x_n$;
- for T_8: $x_{n+6} = x_{n+4} + 2x_{n+3} + 3x_{n+2} + 2x_{n+1} + x_n$.

The proofs for these facts can be shown by induction, using methods similar to those in [2] or [3].

An open problem is the construction of an explicit formula for each of the schemes given above.

References

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