NEGATIVE ORDER GENOCCHI POLYNOMIALS

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1. Introduction

Elsewhere [2], I have investigated the properties of $G_n^{(k)}(x)$, the *Genocchi polynomials of order* k (≥ 0), which were shown to be related to $E_n^{(k)}(x)$, the Euler polynomials of order k, and to $B_n^{(k)}(x)$, the Bernoulli polynomials of order k.

When $k = 1$, we have the Genocchi polynomials of the first order, the simplest polynomials of Genocchi type.

If $x = 0$, the *Genocchi numbers* arise.

Following Nörlund ([4] and [5]), who pioneered the study of $B_n^{(-k)}(x)$ and $E_n^{(-k)}(x)$, the Bernoulli and Euler polynomials, respectively, of negative order, I here offer some of the most important properties of $G_n^{(-k)}(x)$, the *Genocchi polynomials of order* $-k$ $(k > 0,\ n \geq -k)$. So far as I am aware, the material in this contribution represents new information.

The justification for seeking knowledge about the negative order polynomials is stated by Nörlund [4]. After saying that there is advantage in extending to negative order the notion of functions of positive order, Nörlund continues: "*On peut ainsi faire rentrer dans un même cadre des fonctions qui apparaissent jusqu'ici comme distinctes.*" [We can thus combine in one framework functions which up to now appear as distinct.]

Beyond this justification, I feel that the $G_n^{(-k)}(x)$ have a vitality of their own which deserves recognition.

Euler and Bernoulli Polynomials of Negative Order

Nörlund ([4] and [5]) defines the Euler polynomials of negative order $-k$ by

$$(1.1)\qquad \sum_{n=0}^{\infty} \frac{t^n}{n!} E_n^{(-k)}(x|w_1 \ldots w_k) = \frac{(e^{w_1 t} + 1) \cdots (e^{w_k t} + 1)e^{tx}}{2^k}$$

and the Bernoulli polynomials of negative order $-k$ by

$$(1.2)\qquad \sum_{n=0}^{\infty} \frac{t^n}{n!} B_n^{(-k)}(x|w_1 \ldots w_k) = \frac{(e^{w_1 t} - 1) \cdots (e^{w_k t} - 1)e^{tx}}{w_1 \ldots w\ t}.$$

If $w_1 = w_2 = \cdots = w_k = 1$, then (1.1) and (1.2) become

$$(1.1)'\qquad \sum_{n=0}^{\infty} \frac{t^n}{n!} E_n^{(-k)}(x) = \left(\frac{e^t + 1}{2}\right)^k e^{tx}$$

and

$$(1.2)'\qquad \sum_{n=0}^{\infty} \frac{t^n}{n!} B_n^{(-k)}(x) = \left(\frac{e^t - 1}{t}\right)^k e^{tx}.$$

The definition to be given in (2.1) for Genocchi polynomials follows the modified forms (1.1)′ and (1.2)′, though an extension to the patterns in (1.1) and (1.2) could be adopted.

For subsequent comparison with corresponding forms for $G_n^{(-k)}(x)$ $(k = 1, 2, 3, \ldots)$, the first few expressions for $E_n^{(-k)}(x)$ and $B_n^{(-k)}(x)$ are:

(1.3)
$$E_0^{(-k)}(x) = 1$$
$$E_1^{(-k)}(x) = x + \frac{1}{2}k$$
$$E_2^{(-k)}(x) = x^2 + kx + \frac{k(k+1)}{4}$$
$$E_3^{(-k)}(x) = x^3 + \frac{3}{2}kx^2 + \frac{3k(k+1)}{4}x + \frac{k^2(k+3)}{8}$$
$$E_4^{(-k)}(x) = x^4 + 2kx^3 + \frac{3k(k+1)}{2}x^2 + \frac{k^2(k+3)}{2}x + \frac{k(k+1)(k^2+5k-2)}{16}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

and

(1.4)
$$B_0^{(-k)}(x) = 1$$
$$B_1^{(-k)}(x) = x + \frac{k}{2}$$
$$B_2^{(-k)}(x) = x^2 + kx + \frac{k(3k+1)}{12}$$
$$B_3^{(-k)}(x) = x^3 + \frac{3}{2}kx^2 + \frac{k(3k+1)}{4}x + \frac{k^2(k+1)}{8}$$
$$B_4^{(-k)}(x) = x^4 + 2kx^3 + \frac{k(3k+1)}{2}x^2 + \frac{k^2(k+1)}{2}x + \frac{k(15k^3+30k^2+5k-2)}{240}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Putting $k = 1$, we readily derive the table:

(1.5)

	$E_n^{(-1)}(x)$	$B_n^{(-1)}(x)$
$n = 0$	1	1
$n = 1$	$x + \frac{1}{2}$	$x + \frac{1}{2}$
$n = 2$	$x^2 + x + \frac{1}{2}$	$x^2 + x + \frac{1}{3}$
$n = 3$	$x^3 + \frac{3}{2}x^2 + \frac{3}{2}x + \frac{1}{2}$	$x^3 + \frac{3}{2}x^2 + x + \frac{1}{4}$
$n = 4$	$x^4 + 2x^3 + 3x^2 + 2x + \frac{1}{2}$	$x^4 + 2x^3 + 2x^2 + x + \frac{1}{5}$

2. Generalized Genocchi Polynomials of Negative Order

Definition and Basic Properties

Define

$$\sum_{n=-k}^{\infty} G_n^{(-k)}(x)\frac{t^n}{|n|!} = \left(\frac{1+e^t}{2t}\right)^k e^{tx} \qquad (k = 1, 2, 3, \ldots), \tag{2.1}$$

whence

(2.1)′ $G_n^{(-k)}(x)$ is undefined when $n < -k$,

i.e., $n + k \geq 0$ is necessary for the existence of $G_n^{(-k)}(x)$.

Putting $k = 0$ in (2.1) leads to the situation covered in [2] when $k = 0$, so we exclude this repetition.

Calculation in (2.1) gives us the first few Genocchi polynomials:

(2.2)
$$G_{-k}^{(-k)}(x) = |-k|!$$
$$G_{-k+1}^{(-k)}(x) = |-k+1|!\left\{x + \frac{1}{2}k\right\}$$
$$G_{-k+2}^{(-k)}(x) = \frac{|-k+2|!}{2!}\left\{x^2 + kx + \frac{k(k+1)}{2}\right\}$$

$$G_{-k+3}^{(-k)}(x) = \frac{|-k+3|!}{3!}\left\{x^3 + \frac{3k}{2}x^2 + \frac{3k(k+1)}{4}x + \frac{k^2(k+3)}{8}\right\}$$

$$G_{-k+4}^{(-k)}(x) = \frac{|-k+4|!}{4!}\left\{x^4 + 2kx^3 + \frac{3k(k+1)}{2}x^2 + \frac{k^2(k+3)}{2}x + \frac{k(k+1)(k^2+5k-2)}{16}\right\}$$

....

In particular, when $k = 1$:

(2.3) $$G_{-1}^{(-1)}(x) = 1$$

$$G_0^{(-1)}(x) = x + \frac{1}{2}$$

$$G_1^{(-1)}(x) = \frac{1}{2}\left\{x^2 + x + \frac{1}{2}\right\}$$

$$G_2^{(-1)}(x) = \frac{1}{3}\left\{x^3 + \frac{3}{2}x^2 + \frac{3}{2}x + \frac{1}{2}\right\} = \frac{1}{3}\left(x + \frac{1}{2}\right)(x^2 + x + 1)$$

$$G_3^{(-1)}(x) = \frac{1}{4}\left\{x^4 + 2x^3 + 3x^2 + 2x + \frac{1}{2}\right\}$$

$$G_4^{(-1)}(x) = \frac{1}{5}\left\{x^5 + \frac{5}{2}x^4 + 5x^3 + 5x^2 + \frac{5}{2}x + \frac{1}{2}\right\}$$

$$= \frac{1}{5}\left(x + \frac{1}{2}\right)(x^4 + 2x^3 + 4x^2 + 3x + 1)$$

....

The Genocchi numbers $G_n^{(-1)}$ $(n \geq 0)$ thus form the sequence

(2.3)′ $$\frac{1}{2}\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\},$$

while

(2.3)″ $$G_{n-1}^{(-1)} \div G_n^{(-1)} = \frac{n+1}{n} \to 1 \text{ as } n \to \infty.$$

Comparison of (2.1) with (1.1)′ reveals that

(2.4) $$G_n^{(-k)}(x) = \frac{|n|!}{(n+k)!}E_{n+k}^{(-k)}(x).$$

Differentiating both sides of (2.1) w.r.t. x leads to the Appell property [2],

(2.5) $$\frac{dG_n^{(-k)}(x)}{dx} = nG_{n-1}^{(-k)}(x), \quad n + k > 1,\ n > 0,$$

whence

(2.6) $$\frac{d^pG_n^{(-k)}(x)}{dx^p} = n(n-1)\cdots(n-p+1)G_{n-p}^{(-k)}(x), \quad n - p \geq 0,$$

so that, using (2.3), we have

(2.7) $$\frac{d^{n+1}G_n^{(-k)}(x)}{dx^{n+1}} = n!$$

Integration of (2.5) gives (with $n \to n + 1$):

(2.8) $$\int_x^{x+1} G_n^{(-k)}(x)\,dx = \frac{G_{n+1}^{(-k)}(1+x) - G_{n+1}^{(-k)}(x)}{n+1}.$$

Summation Formula

Theorem 1:

$$(2.9)\qquad G_n^{(-k)}(x+y) = \sum_{j=-k}^{n} \frac{|n|!}{(n-j)!\,|j|!} G_j^{(-k)}(x)\, y^{n-j}.$$

Proof:

$$\sum_{n=-k}^{\infty} G_n^{(-k)}(x+y)\frac{t^n}{|n|!} = \left(\frac{1+e^t}{2t}\right)^k e^{tx}e^{ty} = \sum_{r=-k}^{\infty} G_r^{(-k)}(x)\frac{t^r}{|r|!}\sum_{m=0}^{\infty}\frac{y^m t^m}{m!}$$

$$= \sum_{n=-k}^{\infty}\sum_{j=-k}^{n} \frac{|n|!}{(n-j)!\,|j|!} G_j^{(-k)}(x)\, y^{n-j}\frac{t}{|n|!}$$

after rearranging the terms.

Equate coefficients of $t^n/|n|!$ and the result follows.

For example, if $k = n = y = 2$, both sides of the formula (2.9) lead to the expression, also derivable from (2.2),

$$G_2^{(-2)}(x+2) = \frac{1}{12}x^4 + x^3 + 4\tfrac{3}{4}x^2 + 10\tfrac{1}{2}x + 9\tfrac{1}{24}.$$

Furthermore, if $k = 3$, $n = 1$, $x = 0$, and y is replaced by x, then (2.9) gives

$$G_{-1}^{(-3)}(x) = \frac{1}{2}(x^2 + 3x + 3)$$

in conformity with (2.2).

Complementary Arguments

We say that x and $-k - x$ are *complementary arguments*.

Theorem 2:

$$(2.10)\qquad G_n^{(-k)}(-k-x) = (-1)^{n+k}G_n^{(-k)}(x).$$

Proof:

$$\sum_{n=-k}^{\infty} G_n^{(-k)}(-k-x)\frac{t^n}{|n|!} = \left(\frac{1+e^t}{2t}\right)^k e^{(-k-x)t} = (-1)^k\left(\frac{1+e^{-t}}{2(-t)}\right)^k e^{-tx}$$

$$= (-1)^k\sum_{n=-k}^{\infty}(-1)^n G_n^{(-k)}(x)\frac{t^n}{|n|!}$$

$$= (-1)^{n+k}\sum_{n=-k}^{\infty} G_n^{(-k)}(x)\frac{t^n}{|n|!}.$$

Comparison of the coefficients of $t^n/|n|!$ yields the result.

Corollary 1:

$$(2.11)\qquad G_n^{(-k)}(-k-x) = \begin{cases} G_n^{(-k)}(x) & \text{if } k+n \text{ is even,}\\ -G_n^{(-k)}(x) & \text{if } k+n \text{ is odd.}\end{cases}$$

Special cases of interest occur when $x = 0$ and (equivalently) $x = -k$. In either of these instances, consider also $k = 1$.

Corollary 2: In Theorem 2, replace x by $x - (k/2)$. Then

$$(2.12)\qquad G_n^{(-k)}\left(-x-\frac{k}{2}\right) = (-1)^{n+k}G_n^{(-k)}\left(x-\frac{k}{2}\right).$$

If $x = 0$ in Corollary 2 (or $x = -k/2$, $k + n$ odd, in Corollary 1), then

$$(2.13)\qquad G_n^{(-k)}\left(-\frac{k}{2}\right) = 0,\quad k+n \text{ odd},$$

i.e., $G_n^{(-k)}(x)$ has a zero when $x = -k/2$ for $k + n$ odd.

Thus, in (2.2), $G_{-k+\ell}^{(-k)}(x)$ has a zero when $x = -k/2$ for ℓ odd.

Analogue of the Multiplication Theorem

More accurately, this analogue of the multiplication theorem [2] could be called a "division theorem" for negative first order Genocchi polynomials. As in [2], there are two cases to consider, one of which involves $B_n^{(-1)}(x)$. Unfortunately, as for $k > 0$, this theorem does not extend beyond $k = -1$.

Case I: m odd

Theorem 3a:

$$G_n^{(-1)}\left(\frac{x-1}{m}\right) = -m^{-n-1}\sum_{s=-1}^{m-2}(-1)^s G_n^{(-1)}(x+s). \tag{2.14}$$

Proof:

$$\sum_{n=-1}^{\infty}\frac{t^n}{|n|!}\sum_{s=-1}^{m-2}(-1)^s G_n^{(-1)}(x+s) = \sum_{s=-1}^{m-2}\frac{1+e^t}{2t}(-1)^s e^{tx}e^{st}$$

$$= \frac{1+e^t}{2t}e^{tx}(-e^{-t} + 1 - e^t + \cdots + (-1)^{m-2}e^{(m-2)t})$$

$$= \frac{1+e^t}{2t}e^{tx}(-e^{-t})(1 - e^t + e^{2t} - \cdots + (-1)^{m-1}e^{(m-1)t})$$

$$= -\frac{1+e^t}{2t}e^{t(x-1)}\cdot\frac{1+e^{mt}}{1+e^t}, \quad \text{since } m \text{ is odd}$$

$$= -m\left(\frac{1+e^{mt}}{2mt}\right)e^{\frac{mt(x-1)}{m}} = -\sum_{n=-1}^{\infty} m\frac{(mt)^n}{|n|!}G_n^{(-1)}\left(\frac{x-1}{m}\right).$$

Therefore,

$$G_n^{(-1)}\left(\frac{x-1}{m}\right) = -m^{-n-1}\sum_{s=-1}^{m-2}(-1)^s G_n^{(-1)}(x+s), \quad m \text{ odd}.$$

Case II: m even

Theorem 3b:

$$B^{(-1)}\left(\frac{x-1}{m}\right) = 2m^{-n-1}\sum_{s=-1}^{m-2}(-1)^s G_n^{(-1)}(x+s). \tag{2.15}$$

Proof:

$$\sum_{n=-1}^{\infty}\frac{t^n}{|n|!}\sum_{s=-1}^{m-2}(-1)^s G_n^{(-1)}(x+s)$$

$$= \frac{1+e^t}{2t}e^{tx}\cdot -e^{-t}(1 - e^t + e^{2t} - e^{3t} + \cdots + (-1)^{m-1}e^{(m-1)t}), \quad \text{as in Theorem 3a}$$

$$= -\frac{1+e^t}{2t}e^{t(x-1)}\frac{1-e^{mt}}{1+e^t}, \quad \text{since } m \text{ is even}$$

$$= -\frac{e^{t(x-1)}}{2t}(1-e^{mt}) = m\cdot\frac{1}{2}\cdot\frac{e^{mt}-1}{mt}\cdot e^{\frac{mt(x-1)}{m}}$$

$$= \frac{m}{2}\sum_{n=0}^{\infty}\frac{(mt)^n}{n!}B_n^{(-1)}\left(\frac{x-1}{m}\right), \quad \text{on using (1.2)}'$$

$$= \frac{m^{n+1}}{2}\sum_{n=0}^{\infty}\frac{t^n}{n!}B_n^{(-1)}\left(\frac{x-1}{m}\right).$$

Equate corresponding coefficients of $t^n/n!$ and the result follows. It is to be noted that, in the left-hand side summation, $n = -1$ and m even lead to the term

$$\frac{1}{t}\cdot\frac{m}{2}(-1+1) = 0.$$

Relations between Polynomials of Successive Orders

Theorem 4:

$$(2.16)\quad G_n^{(-1)}(x-1)+G_n^{(-1)}(x)=\begin{cases}2nG_{n-1}^{(-2)}(x-1) & n=1,2,3,\ldots,\\ \dfrac{2}{|n-1|!}G_{n-1}^{(-2)}(x-1) & n=-1,0.\end{cases}$$

Proof:

$$\sum_{n=-1}^{\infty}[G_n^{(-1)}(x-1)+G_n^{(-1)}(x)]\frac{t^n}{|n|!}=\left(\frac{1+e^t}{2}\right)e^{tx-t}+\left(\frac{1+e^t}{2}\right)e^{tx}$$

$$=\frac{1+e^t}{2t}e^{tx}(1+e^{-t})=2t\left(\frac{1+e^t}{2t}\right)^2e^{t(x-1)}$$

$$=2G_{-2}^{(-2)}(x-1)\frac{t^{-1}}{|-2|!}+2G_{-1}^{(-2)}(x-1)\frac{t^0}{|-1|!}+\sum_{n=1}^{\infty}2nG_{n-1}^{(-2)}(x-1)\frac{t^n}{n!}$$

Equate coefficients of $t^n/|n|!$ and the result follows.

Clearly, the result can be extended to $G_n^{(-k)}(x)$.

With $x\to x+1$ in Theorem 4, we have

Theorem 5:

$$(2.17)\quad G_n^{(-1)}(1+x)+G_n^{(-1)}(x)=\begin{cases}2nG_{n-1}^{(-2)}(x) & n=1,2,3,\ldots,\\ \dfrac{2}{|n-1|!}G_{n-1}^{(-2)}(x) & n=-1,0,\end{cases}$$

with a straightforward extension to $n=-k$ if desired.

A companion result is

Theorem 6:

$$(2.18)\quad G_n^{(-1)}(1+x)-G_n^{(-1)}(x)=2^nB_n^{(-1)}\left(\frac{x}{2}\right),\qquad (n\ge 0).$$

Proof:

$$\left[\sum_{n=0}^{\infty}G_n^{(-1)}(1+x)-G_n^{(-1)}(x)\right]\frac{t^n}{n!}=\left(\frac{1+e^t}{2t}\right)(e^t-1)e^{tx}$$

$$=\frac{e^{2t}-1}{2t}e^{2t\cdot\frac{x}{2}}$$

$$=2^n\sum_{n=0}^{\infty}B_n^{(-1)}\left(\frac{x}{2}\right)\frac{t^n}{n!},\quad\text{on using (1.2)}',$$

from which the formula follows.

To generalize Theorem 6, we need to expand $(e^t-1)^k$. After suitable algebraic manipulation, it ensues as in the proof of Theorem 6 that

$$(2.19)\quad \sum_{j=0}^{k}(-1)^{j-1}\binom{k}{j}G_n^{(-k)}(j+x)=(-1)^{k+1}2^nB_n^{(-k)}\left(\frac{x}{2}\right)\qquad (n\ge 0).$$

Theorem 7:

$$(2.20)\quad (n+1)G_n^{(-1)}(x)=n(x+1)G_{n-1}^{(-1)}(x)-\frac{1}{2}G_n^{(0)}(x)\qquad (n\ge 1).$$

Proof: Differentiate both sides of (2.1) for $k=1$ w.r.t. t partially, and then multiply by t. It follows that

$$-\frac{1}{t}+\sum_{n=1}^{\infty}G_n^{(-1)}(x)\frac{nt^n}{n!}=\left(\frac{1+e^t}{2}\right)e^{tx}\cdot xt+\frac{e^{tx}}{2}\left(\frac{te^t-(1+e^t)}{t}\right)$$

$$=\left(\frac{1+e^t}{2t}\right)e^{tx}\cdot xt-\left(\frac{1+e^t}{2t}\right)e^{tx}+\frac{e^{tx}}{2t}t(1+e^t-1)$$

$$=\left(\frac{1+e^t}{2t}\right)e^{tx}t(x+1)-\left(\frac{1+e^t}{2t}\right)e^{tx}-\frac{e^{tx}}{2}.$$

Equate coefficients of $t^n/n!$ and the result follows. Observe (see [2]) that $G_n^{(0)}(x) = x^n$.

The $n = 0$ term, being a constant, does not contribute to the summation on differentiation w.r.t. t partially.

Proceeding in the same manner, we may establish the generalization

$$(2.21)\quad (n + k)G_n^{(-k)}(x) = n(k + x)G_{n-1}^{(-k)}(x) - \frac{k}{2}G_n^{-(k-1)}(x) \qquad (n \geq 1).$$

In particular, when $k = 2$, the left-hand side of the first line of the proof in Theorem 7 (after partial differentiation and multiplication by t) becomes

$$-\frac{2}{t^2} - \frac{(1 + x)}{t} + 0 + \sum_{n=1}^{\infty} G_n^{(-2)}(x)\frac{nt}{n!},$$

since the $n = 0$ term does not contribute, being a constant as far as partial differentiation w.r.t. t is concerned.

$G_n^{(-k)}(x)$ in Terms of $G_m^{(-1)}(f(x))$

Adopting a different technique, we are enabled to derive formulas connecting $G_n^{(-k)}(x)$ with negative first order Genocchi polynomials of appropriate functions $f(x)$ of x. When $k = 2, 3$, we have

Theorem 8: If $n \geq 0$,

$$(2.22)\quad \begin{aligned} 2(n + 1)G_n^{(-2)}(x) &= 2\left\{2^{n+1}G_{n+1}^{(-1)}\left(\frac{x}{2}\right) + G_{n+1}^{(-1)}(x)\right\} - \frac{x^{n+2}}{n + 2}, \\ 4(n + 2)(n + 1)G_n^{(-3)}(x) &= 3\left\{3^{n+1}G_{n+2}^{(-1)}\left(\frac{x}{3}\right) + G_{n+2}^{(-1)}(x + 1)\right\}. \end{aligned}$$

Proof: Consider

$$(2.23)\quad \left(\frac{1 + e^t}{2t}\right)^2 e^{tx} = \frac{2}{2t}\left(\frac{1 + e^{2t}}{2 \cdot 2t}\right)e^{2t\cdot\frac{x}{2}} + \frac{2}{2t}\left(\frac{1 + e^t}{2t}\right)e^{tx} - \frac{e^{tx}}{2t^2}$$

and

$$(2.24)\quad \left(\frac{1 + e^t}{2t}\right)^3 e^{tx} = \frac{3}{4t^2}\left(\frac{1 + e^{3t}}{2 \cdot 3t}\right)e^{3t\cdot\frac{x}{3}} + \frac{3}{4t^2}\left(\frac{1 + e^t}{2t}\right)^{t(x+1)}.$$

Equate coefficients of $t^n/n!$ and the results follow. ($x^{n+2} = G_{n+2}^{(0)}(x)$ by [2].)

Determination of the somewhat complicated extensions of (2.22) for general k is left to the curiosity of the reader. Depending on the parity of k, we will obtain two separate expressions in the generalization. Nevertheless, there is a unifying principle in the proof, namely, the grouping of pairs of appropriate terms; when k is even, there will be additionally a single unpaired term.

Similar kinds of results may be obtained for $E_n^{(-k)}(x)$ and $B_n^{(-k)}(x)$ on using (1.1)′ and (1.2)′. However, in the case of Bernoulli polynomials we remark that, for k even, $B_n^{(-k)}(x)$ is expandable in terms of Genocchi polynomials.

$G_n^{(-1)}(x)$ in Terms of $G_m^{(-1)}\left(\frac{1}{2}\right)$

Theorem 9:

$$(2.25)\quad G_n^{(-1)}(x) = \sum_{r=-1}^{n} \frac{|n|!}{(n - r)!\,|r|!}G_r^{(-1)}\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right)^{n-r}.$$

Proof:

$$\sum_{n=-1}^{\infty} G_n^{(-1)}(x)\frac{t^n}{|n|!} = \left(\frac{1 + e^t}{2t}\right)e^{tx} = \left(\frac{1 + e^t}{2t}\right)e^{\frac{t}{2}} \cdot e^{\left(x-\frac{1}{2}\right)t}$$

$$= \sum_{s=-1}^{\infty} \frac{G_{-1}^{(-1)}\left(\frac{1}{2}\right)}{|-1|!t} \cdot \frac{\left(x - \frac{1}{2}\right)^{s+1} t^{s+1}}{(s+1)!} + \left\{\sum_{r=0}^{\infty} G^{(-1)}\left(\frac{1}{2}\right)\frac{t}{r!}\right\}\left\{\sum_{m=0}^{\infty}\left(x - \frac{1}{2}\right)^m \frac{t^m}{m!}\right\}.$$

Application of Cauchy's multiplication of power series and comparison of coefficients of $t^n/n!$ yield the desired result.

Sums of Products

What happens if we square both sides of (2.1)? Clearly,

$$(2.26)\quad \left(\sum_{n=-1}^{\infty} G_n^{(-1)}(x)\frac{t^n}{|n|!}\right)\left(\sum_{n=-1}^{\infty} G_n^{(-1)}(x)\frac{t^n}{|n|!}\right) = \left(\frac{1+e^t}{2}\right)^2 e^{t.2x}$$
$$= \sum_{n=-2}^{\infty} G_n^{(-2)}(2x)\frac{t^n}{|n|!}.$$

Comparison of coefficients of $t^n/|n|!$ yields a set of sums of products, expressible in general form as

$$(2.27)\quad G_n^{(-2)}(2x) = \begin{cases} 2\displaystyle\sum_{j=-1}^{\left[\frac{n}{2}\right]} G_j^{(-1)}(x)\frac{G_{n-j}^{(-1)}(x)}{|n-j|!} & n \text{ odd,} \\ 2\displaystyle\sum_{j=-1}^{\left[\frac{n-1}{2}\right]} G_j^{(-1)}(x)\frac{G_{n-j}^{(-1)}(x)}{|n-j|!} + G_{n/2}^{(-1)}(x) & n \text{ even.} \end{cases}$$

Furthermore, if we replace t by $-t$ in one of the infinite sums in (2.26), we find

$$(2.28)\quad \left(\sum_{n=-1}^{\infty} G_n^{(-1)}(x)\frac{t^n}{|n|!}\right)\left(\sum_{n=-1}^{\infty} G_n^{(-1)}(x)\frac{(-t)^n}{|n|!}\right) = -\left(\frac{1+e^t}{2t}\right)^2 e^{-t}$$
$$= -\sum_{n=-2}^{\infty} G_n^{(-2)}(-1)\frac{t^n}{|n|!},$$

leading to formulas for $G_n^{(-2)}(-1)$ similar to those in (2.27). Observe that

$G_n^{(-2)}(-1) = 0$ when n is odd, by (2.13).

Putting $x = -1/2$ in (2.27), we also obtain formulas for $G_n^{(-2)}(-1)$ in terms of $G_m^{(-1)}(-1/2)$.

Interested readers may wish to extend the above theory to unspecified k in $G_n^{(-k)}(x)$. Additionally, one may determine results corresponding to those in (2.27) for Euler and Bernoulli polynomials.

3. Miscellaneous Theorems

Use of Boole's Theorem

For a polynomial $P(x)$, *Boole's theorem* states that

$$P(x+y) = \nabla P(x) + E_1(y)\nabla P'(x) + \frac{1}{2!}E_2(y)\nabla P''(x) + \frac{1}{3!}E_3(y)\nabla P'''(x) + \cdots,$$

where the symbol ∇ ('nabla') represents the operation of the mean of the function (see [2]) and $E_i(x)$ ($i = 1, 2, 3, \ldots$) are the Euler polynomials $E_i^{(1)}(x)$ obtained from (1.3) by replacing k by -1. Prime superscripts signify differentiation w.r.t. x.

Now

$$\nabla G_n^{(-1)}(x) = \frac{1}{2}(G_n^{(-1)}(1+x) + G_n^{(-1)}(x)) \qquad \text{by the definition of } \nabla$$

$$= \begin{cases} nG_{n-1}^{(-2)}(x) & (n = 1, 2, 3, \ldots) \\ \dfrac{1}{|n-1|!}G_{n-1}^{(-2)}(x) & (n = -1, 0) \end{cases} \quad \text{by Theorem 5.}$$

Put $y = 0$ in Boole's theorem and take $P(x) = G_n^{(-1)}(x)$.

Then Boole's theorem becomes, for $n > 0$ (2.5),

$$G_n^{(-1)}(x) = \nabla G_n^{(-1)}(x) + E_1(0)\nabla G_n^{(-1)\prime}(x) + \frac{1}{2!}E_2(0)\nabla G_n^{(-1)\prime\prime}(x) + \cdots,$$

that is,

Theorem 10: When $n = 1, 2, 3, \ldots,$

$$(3.1) \qquad G_n^{(-1)} = nG_{n-1}^{(-2)}(x) + E_1(0)\cdot nG_{n-1}^{(-2)\prime}(x) + \frac{1}{2!}E_2(0)\cdot nG_{n-1}^{(-2)\prime\prime}(x) + \cdots .$$

For example, if $n = 2$, the right-hand side reduces to

$$\frac{1}{3}\left(x^3 + \frac{3}{2}x^2 + \frac{3}{2}x + \frac{1}{2}\right) \quad [= G_2^{(-1)}(x) \text{ as in (2.3)}].$$

Genocchi Polynomials in Terms of Bernoulli Polynomials

The *Euler-Maclaurin theorem* (see [3]) states, in the case of polynomials $G_n^{(-1)}(x)$, that

$$G_n^{(-1)\prime}(x) = \Delta G_n^{(-1)}(0) + B_1(x)\Delta G_n^{(-1)\prime}(x) + \frac{B_2(x)}{2!}\Delta G_n^{(-1)\prime\prime}(0) + \cdots,$$

where $B_i(x)$ $(i = 1, 2, 3, \ldots)$ are the Bernoulli polynomials $B_i^{(1)}(x)$ obtained from (1.4) by replacing k by -1 and Δ is the symbol for the operation of taking the difference.

Now, by (2.5),

$$G_n^{(-1)\prime}(x) = nG_{n-1}^{(-1)}(x) \qquad (n > 0)$$

and, by the definition of Δ,

$$(3.2) \qquad \Delta G_n^{(-1)}(x) = G_n^{(-1)}(1 + x) - G_n^{(-1)}(x)$$

$$= 2^n B_n^{(-1)}\left(\frac{x}{2}\right) \qquad \text{by Theorem 6 } (n \ge 0).$$

Then, by (2.5) and (3.2), the Euler-Maclaurin theorem leads to

Theorem 11:

$$(3.3) \qquad nG_{n-1}^{(-1)}(x) = 2^n\left\{B_n^{(-1)}(0) + B_1(x)B_n^{(-1)\prime}(0) + \frac{B_2(x)}{2!}B_n^{(-1)\prime\prime}(0) + \cdots\right\} \quad (n \ge 1).$$

When $n = 3$, the theorem reduces to

$$3G_2^{(-1)}(x) = x^3 + \frac{3}{2}x^2 + \frac{3x}{2} + \frac{1}{2},$$

which is true by (2.3). Theorem 11 enables us to display $G_n^{(-1)}(x)$ entirely by means of Bernoulli expressions. Both Theorems 10 and 11 (for $k = 1$) may be extended to cover the case when k is general.

Some 'Hybrid' Products

Let us write

$$(3.4)\quad \begin{cases} G \equiv \sum_{n=0}^{\infty} G_n^{(1)}(x)\dfrac{t^n}{n!}, & G_- \equiv \sum_{n=0}^{\infty} G_n^{(1)}(x)\dfrac{(-t)^n}{n!}, \\ G^* \equiv \sum_{n=-1}^{\infty} G_n^{(-1)}(x)\dfrac{t^n}{|n|!}, & G_-^* \equiv \sum_{n=-1}^{\infty} G_n^{(-1)}(x)\dfrac{(-t)^n}{|n|!} \end{cases}$$

where G is as defined in [2], G^* refers to (2.1) when $k = 1$, and G_-, G_-^* are obtained from G, G^*, respectively, by replacing t by $-t$. Corresponding symbolism E, ..., E_-^*, B, ..., B_-^* relates to Euler and Bernoulli polynomials, where E and B are also defined in [2].

Then, by [2] and (2.1)

$$(3.5)\quad GG^* = e^{2tx}$$

and

$$(3.6)\quad GG_-^* = -e^{-t}.$$

Equating appropriate coefficients yields the hybrid results

$$(3.7)\quad \sum_{j=1}^{n+1}\left(\frac{G_j^{(1)}(x)}{j!}\cdot\frac{G_{n-j}^{(-1)}(x)}{|n-j|!}\right) = \frac{(2x)^n}{n!}$$

and

$$(3.8)\quad \sum_{j=1}^{n+1}\left(\frac{G_n^{(1)}(x)}{j!}\quad\frac{(-1)^{n-j}G_{n-j}^{(-1)}(x)}{|n-j|!}\right) = \frac{(-1)^{n-1}}{|n-1|!}.$$

Similarly,

$$(3.9)\quad G_-G^* = -e^t = (GG_-^*)^{-1}$$

and

$$(3.10)\quad G_-G_-^* = e^{-2tx} = (GG^*)^{-1},$$

yielding results corresponding to (3.7) and (3.8). The case $G^*G_{-1}^*$ has been covered in (2.28). In addition,

$$G_-^*G_-^* = \left(\frac{1+e^t}{2t}\right)^2 e^{-t(2x+2)}$$

gives the summation (2.1) for $G_n^{(-2)}\{-(2x+2)\}$.

Moreover,

$$(3.11)\quad \begin{cases} EE^* = BB^* = e^{2tx} \\ E_-E^* = B_-B^* = e^t \\ G^*E = \dfrac{1}{t}e^{2tx} \\ GE^* = te^{2tx} \\ G^*E^* = t\left(\dfrac{1+e^t}{2t}\right)^2 e^{t\cdot 2x} \\ G^*B^* = \dfrac{1}{t}\,\dfrac{e^{2t}-1}{2t}e^{2t\cdot\frac{x}{2}} \\ B_-^*E^* = \dfrac{e^{2t}-1}{2t}e^{2t\left(-\frac{1}{2}\right)} \end{cases}$$

for example, among a variety of possible products. The last three equations in (3.11) give the summations (2.1) and (1.2)′ for

$$G_{n-1}^{(-2)}(2x),\quad B_{n+1}^{(-1)}\left(\frac{x}{2}\right),\ \text{and}\ B_n^{(-1)}\left(-\frac{1}{2}\right),$$

respectively.

Our theory may be extended to values of $k > 1$.

Products of powers of the G, E, and B symbols give rise to an immense number of identities, for example

$$(3.12)\quad \begin{cases} GG_-G^*G_-^*{}_1 = 1, \\ GE(E^*)^2 = te^{4tx}, \\ G^3G_-^2(G^*)^2B_-B^*(E_-^*)^3 = t^3. \end{cases}$$

To avoid tedium, we leave the challenge of exploring such possibilities, which may be continued almost *ad infinitum, ad nauseam*!, to the ingenuity and perseverance of the reader.

4. Differential Equations

Descending Diagonal Functions

Arrange the $G_n^{(-1)}(x)$ in (2.3) according to the following pattern:

$$(4.1)\quad \begin{aligned} G_{-1}^{(-1)}(x) &= G_{-1}^{(-1)} \\ G_0^{(-1)}(x) &= G_0^{(-1)} + xG_{-1}^{(-1)} \\ G_1^{(-1)}(x) &= G_1^{(-1)} + xG_0^{(-1)} + \frac{1}{2}x^2G_{-1}^{(-1)} \\ G_2^{(-1)}(x) &= G_2^{(-1)} + 2xG_1^{(-1)} + x^2G_0^{(-1)} + \frac{1}{3}x^3G_{-1}^{(-1)} \\ G_3^{(-1)}(x) &= G_3^{(-1)} + 3xG_2^{(-1)} + 3x^2G_1^{(-1)} + x^3G_0^{(-1)} + \frac{1}{4}x^4G_{-1}^{(-1)} \\ G_4^{(-1)}(x) &= G_4^{(-1)} + 4xG_3^{(-1)} + 6x^2G_2^{(-1)} + 4x^3G_1^{(-1)} + x^4G_0^{(-1)} + \frac{1}{5}x^5G_{-1}^{(-1)} \\ &\cdots\cdots\cdots \end{aligned}$$

in which

$$(4.2)\quad G_n^{(-1)}(x) = \sum_{j=-1}^{n} \frac{|n|!}{(n-j)!\,|j|!}G_j^{(-1)}x^{n-j}$$

as in [2] for $G_n^{(1)}(x)$.

Imagine now that the terms are considered to lie in an infinite set of downward slanting "parallel lines" to form the following set of *descending diagonal functions* $\{g_n^{(-1)}(x)\}$ $(n=-1, 0, 1, 2, \ldots)$ and their generating functions $(|x| < 1)$:

$$(4.3)\quad \begin{aligned} g_{-1}^{(-1)}(x) &= G_{-1}^{(-1)}\left(1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots\right) = G_{-1}^{(-1)}(1 - \log(1 - x)) \\ g_0^{(-1)}(x) &= G_0^{(-1)}(1 + x + x^2 + x^3 + x^4 + \cdots) = G_0^{(-1)}(1 - x)^{-1} \\ g_1^{(-1)}(x) &= G_1^{(-1)}(1 + 2x + 3x^2 + 4x^3 + \cdots) = G_1^{(-1)}(1 - x)^{-2} \\ g_2^{(-1)}(x) &= G_2^{(-1)}(1 + 3x + 6x^2 + \cdots) = G_2^{(-1)}(1 - x)^{-3} \\ &\cdots\cdots\cdots \end{aligned}$$

with, generally, as in [2] for $G_n^{(1)}(x)$,

$$(4.4)\quad g_n^{(-1)}(x) = G_n^{(-1)}(1 - x)^{-(n+1)}.$$

Note that

$$(4.5)\quad \begin{cases} g_n^{(-1)}(x) = G_n^{(-1)} \displaystyle\sum_{j=0}^{\infty} \binom{n+j}{j} x^j & n \geq 0 \\ g_n^{(-1)}(0) = G_n^{(-1)} & \\ g_n^{(-1)}\left(\frac{1}{2}\right) = 2^{n+1} G_n^{(-1)} & n \geq 0 \\ \qquad\quad = (1 + \log 2) G_{-1}^{(-1)} & n = -1 \\ g_n^{(-1)}(1) \text{ is not defined.} & \end{cases}$$

Write

$$(4.6)\quad D \equiv D(x, y) = \sum_{n=1}^{\infty} g_{n-1}^{(-1)}(x) y^{n-1} = \sum_{n=1}^{\infty} G_{n-1}^{(-1)} (1 - x)^{-n} y^{n-1}$$

whence

$$(4.7)\quad ny\frac{\partial D}{\partial y} - (n - 1)(1 - x)\frac{\partial D}{\partial x} = 0,$$

while, from (4.5),

$$(4.8)\quad (1 - x)\frac{dg_n^{(-1)}(x)}{dx} = (n + 1) g_n^{(-1)}(x).$$

Observe in (4.6) that $g_{-1}^{(-1)}(x)$ has been omitted.

Reverting now to (4.2), we may easily generalize this formula by replacing -1 by $-k$ (three times). For what follows, the reader may find it helpful to construct a partial table like (4.1) from (2.2). An analysis of the cases $k = 2, 3, \ldots$ then discloses the interesting nexus:

$$(4.9)\quad \frac{g_n^{(-1)}(x)}{G_n^{(-1)}} = \frac{g_n^{(-2)}(x)}{G_n^{(-2)}} = \frac{g_n^{(-3)}(x)}{G_n^{(-3)}} = \cdots \begin{cases} = (1 - x)^{-n+1} & (n = 0, 1, 2, \ldots) \\ = 1 - \log(1 - x) & (n = -1) \end{cases}$$

When $n < -1$, there is no such simple pattern as in (4.9) [though, exceptionally, $g_{-2}^{(-2)}(x)$ is expressible in terms of $g_{-1}^{(-1)}(x)$]. This unstructured situation results from the somewhat wayward behavior, as k varies, of

$$(4.10)\quad g_{-k}^{(-k)}(x) = G_{-k}^{(-k)}\left\{1 + \frac{1}{|-k|!}\left(|-k + 1|!x + \frac{|-k + 2|!}{2!}x^2 + \frac{|-k + 3|!}{3!}x^3 + \cdots\right)\right\}$$

which is aberrant on account of the unusual presence of modulus factorials.

The repetitive nature of the $g_n^{(-k)}(x)$ is understood if we examine successive levels in the layout of

$$G_{-k}^{(-k)}(x), \quad G_{-k+1}^{(-k)}(x), \quad G_{-k+2}^{(-k)}(x), \quad \ldots$$

corresponding to (4.1).

Consider, for example, the coefficients of x in $G_{-k+3}^{(-k)}(x)$ and $G_{-k+4}^{(-k)}(x)$, i.e.,

$$\frac{|-k + 3|!}{|-k + 2|!} G_{-k+2}^{(-k)} \quad \text{and} \quad \frac{|-k + 4|!}{|-k + 3|!} G_{-k+3}^{(-k)},$$

respectively. Substituting $k = 2$ in the first case and $k = 3$ in the second, we have immediately $1 \cdot G_0^{(-2)}$ and $1 \cdot G_0^{(-3)}$, i.e., the coefficient 1 is repeated.

Rising Diagonal Functions

Concentrate next on the infinite set of upward slanting "parallel lines" which form the following *rising diagonal functions*:

$$
\begin{aligned}
(4.11)\quad h_{-1}^{(-1)}(x) &= G_{-1}^{(-1)}\\
h_{0}^{(-1)}(x) &= G_{0}^{(-1)}\\
h_{1}^{(-1)}(x) &= xG_{-1}^{(-1)} + G_{1}^{(-1)}\\
h_{2}^{(-1)}(x) &= xG_{0}^{(-1)} + G_{2}^{(-1)}\\
h_{3}^{(-1)}(x) &= \tfrac{1}{2}x^2G_{-1}^{(-1)} + 2xG_{1}^{(-1)} + G_{3}^{(-1)}\\
h_{4}^{(-1)}(x) &= x^2G_{0}^{(-1)} + 3xG_{2}^{(-1)} + G_{4}^{(-1)}\\
h_{5}^{(-1)}(x) &= \tfrac{1}{3}x^3G_{-1}^{(-1)} + 3x^2G_{1}^{(-1)} + 4xG_{3}^{(-1)} + G_{5}^{(-1)}\\
h_{6}^{(-1)}(x) &= x^3G_{0}^{(-1)} + 6x^2G_{2}^{(-1)} + 5xG_{4}^{(-1)} + G_{6}^{(-1)}\\
h_{7}^{(-1)}(x) &= \tfrac{1}{4}x^4G_{-1}^{(-1)} + 4x^3G_{1}^{(-1)} + 10x^2G_{3}^{(-1)} + 6xG_{5}^{(-1)} + G_{7}^{(-1)}\\
h_{8}^{(-1)}(x) &= x^4G_{0}^{(-1)} + 10x^3G_{2}^{(-1)} + 15x^2G_{4}^{(-1)} + 7xG_{6}^{(-1)} + G_{8}^{(-1)}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

Generally,

$$(4.12)\quad h_n^{(-1)}(x) = \sum_{j=0}^{\left[\frac{n+1}{2}\right]} \frac{|n-j|!}{j!\,|n-2j|!} G_{n-2j}^{(-1)} x^j.$$

Clearly,

$$(4.13)\quad h_n^{(-1)}(0) = G_n^{(-1)} = g_n^{(-1)}(0).$$

Consider

$$
\begin{aligned}
(4.14)\quad R \equiv R(x,\, y) &= \sum_{n=1}^{\infty} h_{n-1}^{(-1)}(x) y^{n-1}\\
&= (1 - xy^2)^{-1}G_0^{(-1)} + y(1 - xy^2)^{-2}G_1^{(-1)}\\
&\qquad + y^2(1 - xy\ )^{-3}G_2^{(-1)} + \cdots .
\end{aligned}
$$

Writing

$$(4.15)\quad \psi \equiv (1 - xy^2)^{-2}G_0^{(-1)} + y(1 - xy^2)^{-3}G_1^{(-1)} + y^2(1 - xy^2)^{-4}G_2^{(-1)} + \cdots$$

and

$$(4.16)\quad \phi \equiv (1 - xy^2)^{-2}G_1^{(-1)} + 2y(1 - xy^2)^{-3}G_2^{(-1)} + 3y^2(1 - xy^2)^{-4}G_3^{(-1)} + \cdots$$

we readily obtain, as in [2], the partial differential equations

$$(4.17)\quad \frac{\partial R}{\partial x} = y^2\psi$$

and

$$(4.18)\quad \frac{\partial R}{\partial y} = 2xy\psi + \phi,$$

leading to

$$(4.19)\quad \frac{\partial \phi}{\partial x} = y^2\,\frac{\partial \psi}{\partial y} - 2xy\,\frac{\partial \psi}{\partial x}$$

on partially differentiating (4.17) w.r.t. y and (4.18) w.r.t. x and then apply-ign Bernoulli's theorem:

$$\frac{\partial^2 R}{\partial x \partial y} = \frac{\partial^2 R}{\partial y \partial x}.$$

Generally,

$$(4.20)\quad h_n^{(-k)}(x) = \sum_{j=0}^{\left[\frac{n+k}{2}\right]} \frac{|n-j|!}{j!\,|n-2j|!} G_{n-2j}^{(-k)} x^j,$$

i.e., -1 in (4.12) has been replaced by $-k$ (three times), and an extended theory for differential equations may be pursued corresponding to that given in [2]. Observe that, whereas in (4.20) the number $G_{-1}^{(-1)}$ has been omitted, in the general case, the numbers $G_{-1}^{(-k)}$, $G_{-2}^{(-k)}$, ..., $G_{-k}^{(-k)}$ will be missing.

5. Concluding Remarks

Many other properties of $G_n^{(-k)}(x)$ may be developed, but it is hoped that this exposition will give a flavor of the basic ingredients of the mixture. Further extensions could, for instance, involve relationships with $B_n^{(-k)}(x)$ and $E_n^{(-k)}(x)$. As a guide to the possibilities, one might consult [2] for corresponding material relating to $G_n^{(-k)}(x)$, e.g., graphs, and for appropriate references.

In treating $G_n^{(-k)}(x)$, there is the obvious choice of deciding whether or not to exclude the cases $n = -k, -k + 1, \ldots, -1$. Inclusion of these values does add to complications in the theory. Without them, one can sometimes proceed from results in [2] for $k \geq 0$ to those established here, simply by replacing k by $-k$. This situation gives the continuity and unity mentioned by Nörlund (for Euler and Bernoulli polynomials) in the French quote in the Introduction.

Consideration of negative values of n in $G_n^{(-k)}(x)$ adds much to the completeness of the theory and, despite the difficulties involved, enhances the enjoyment of the work.

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