A MATRIX METHOD TO SOLVE LINEAR RECURRENCES WITH CONSTANT COEFFICIENTS*

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In this paper we provide a matrix method to solve linear recurrences with constant coefficients.

Consider the linear recurrence relation with constant coefficients

$$(1)\quad \begin{cases} u_{n+k} = \alpha_1 u_{n+k-1} + \alpha_2 u_{n+k-2} + \cdots + \alpha_k u_n + b_n & (1.1) \\ u_0 = c_0,\ u_1 = c_1,\ \ldots,\ u_{k-1} = c_{k-1} & (1.2) \end{cases}$$

where α_i and c_i are constants ($i = 0, 1, 2, \ldots, k$) and where $\langle b_n \rangle_{n \in N}$ is a given sequence.

In order to solve this recurrence relation generally, we first find the general solution $\langle \tilde{u}_m \rangle_{m \in N}$ of the corresponding homogeneous relation

$$(2)\quad \begin{cases} u_{n+k} = \alpha_1 u_{n+k-1} + \alpha_2 u_{n+k-2} + \cdots + \alpha_k u_n \\ u_0 = c_0,\ u_1 = c_1,\ \ldots,\ u_{k-1} = c_{k-1} \end{cases}$$

and then find a particular solution $\langle u'_m \rangle_{m \in N}$ of (1) satisfying the initial conditions. Then $\langle \tilde{u}_m + u'_m \rangle_{m \in N}$ is a solution of (1).

The general method (see [1]) for solving recurrence (2) requires, as a first step, solving the corresponding characteristic equation

$$(3)\quad \lambda^k - \alpha_1 \lambda^{k-1} - \alpha_2 \lambda^{k-2} - \cdots - \alpha_k = 0.$$

Generally, when $k \geq 3$, it is rather difficult to find the roots λ_i of (3).

Now we construct a matrix A such that (3) is the characteristic equation of A, and then obtain the general solution of (1) from A^m.

Let A be the $k \times k$ companion matrix of the polynomial of (3):

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ \alpha_k & \alpha_{k-1} & \alpha_{k-2} & & \alpha_2 & \alpha_1 \end{bmatrix}.$$

Then the characteristic equation of A is (3) and, by the Hamilton-Cayley theorem,

$$(4)\quad A^k - \alpha_1 A^{k-1} - \alpha_2 A^{k-2} - \cdots - \alpha_k I = 0.$$

Consider the following $k \times 1$ matrices:

$$C = (c_0, c_1, \ldots, c_{k-1})^t,\ B_j = (0, 0, \ldots, 0, b_j)^t,\ j = 0, 1, \ldots .$$

Let

$$(5)\quad A^m C + A^{m-1} B_0 + A^{m-2} B_1 + \cdots + A^{k-1} B_{m-k} = (a^{(m)}, \ldots)^t.$$

We will prove that $\langle a^{(m)} \rangle_{m \in N}$ satisfies (1). By equation (4),

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Hence,

$$A^mC = \sum_{i=1}^{k}\alpha_i A^{m-i}C,\; A^{m-j-1}B_j = \sum_{i=1}^{k}\alpha_i A^{m-j-1-i}B_j,\; j = 0, 1, 2, \ldots .$$

(6)
$$\begin{aligned}(a^{(n+k)}, \ldots)^t &= A^{n+k}C + A^{n+k-1}B_0 + A^{n+k-2}B_1 + \cdots + A^kB_{n-1} + A^{k-1}B_n\\
&= \sum_{i=1}^{k}\alpha_i A^{n+k-i}C + \sum_{i=1}^{k}\alpha_i A^{n+k-1-i}B_0 + \cdots + \sum_{i=1}^{k}\alpha_i A^{k-i}B_{n-1} + A^{k-1}B_n\\
&= \alpha_1\left(A^{n+k-1}C + \sum_{i=1}^{n}A^{n+k-1-i}B_{i-1}\right) + \alpha_2\left(A^{n+k-2}C + \sum_{i=1}^{n-1}A^{n+k-2-i}B_{i-1}\right)\\
&\quad + \sum_{i=2}^{k}\alpha_i A^{k-i}B_{n-1} + \alpha_3\left(A^{n+k-3}C + \sum_{i=1}^{n-2}A^{n+k-3-i}B_{i-1}\right)\\
&\quad + \sum_{i=3}^{k}\alpha_i A^{k+1-i}B_{n-2} + \cdots + \alpha_k\left(A^nC + \sum_{i=1}^{n-k+1}A^{n-i}B_{i-1}\right)\\
&\quad + \alpha_k A^{k-2}B_{n-k+1} + A^{k-1}B_n.\end{aligned}$$

Since

$$A^i = \begin{bmatrix} 0 \ldots 0 \overset{(i+1)}{1} 0 \ldots 0 \\ \vdots \qquad\qquad \vdots \end{bmatrix}, \quad i = 0, 1, 2, \ldots, k-1,$$

$$A^iB_j = (0, \ldots)^t, \text{ when } 0 \le i \le k-2,$$

and

$$A^{k-1}B_n = (b_n, \ldots)^t.$$

Then, from (6), we have:

$$\begin{aligned}(a^{(n+k)}, \ldots)^t &= \alpha_1(a^{(n+k-1)}, \ldots)^t\\
&\quad + \alpha_2(a^{(n+k-2)}, \ldots)^t + (0, \ldots)^t\\
&\quad + \alpha_3(a^{(n+k-3)}, \ldots)^t + (0, \ldots)^t\\
&\quad \vdots\\
&\quad + \alpha_k(a^{(n)}, \ldots)^t + (0, \ldots)^t + (b_n, \ldots)^t.\end{aligned}$$

This is

$$a^{(n+k)} = \alpha_1 a^{(n+k-1)} + \alpha_2 a^{(n+k-2)} + \cdots + \alpha_k a^{(n)} + b_n,$$

and (1.1) is satisfied.

By (5),

$$\begin{aligned}(a^{(0)}, \ldots)^t &= A^0C = (c_0, \ldots)^t\\
(a^{(1)}, \ldots)^t &= AC = (c_1, \ldots)^t\\
&\vdots\\
(a^{(k-1)}, \ldots)^t &= A^{k-1}C = (c_{k-1}, \ldots)^t,\end{aligned}$$

that is, $a^{(i)} = c_i$, $i = 0, 1, 2, \ldots, k-1$, and (1.2) also holds. Thus,

(7)
$$\langle u_m\rangle_{m\in N} = \langle a^{(m)}\rangle_{m\in N}$$

is a solution of (1). Now we find a combinatorial expression for $a^{(m)}$. From formula (5),

(8)
$$a^{(m)} = c_0a_{11}^{(m)} + c_1a_{12}^{(m)} + c_2a_{13}^{(m)} + \cdots + c_{k-1}a_{1k}^{(m)} + b_0a_{1k}^{(m-1)} + b_1a_{1k}^{(m-2)} + \cdots + b_{m-k}a_{1k}^{(k-1)}.$$

We consider the associated directed graph D of A with weights $\alpha_1, \alpha_2, \ldots, \alpha_k$ as drawn in Figure 1.

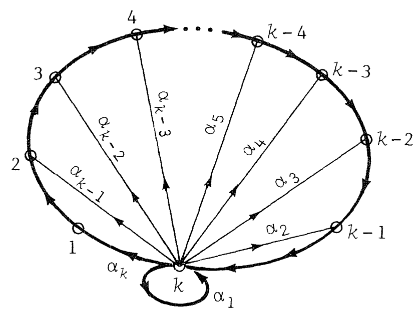

Figure 1

The Associated Digraph D of A
(Arcs with no assigned weight have weight 1.)

The definition of D is given as follows. If $A = [a_{ij}]$, then D is the digraph in which there is an arc (i, j) with weight a_{ij} from i to j if and only if $a_{ij} \neq 0$ $(i, j = 1, \ldots, n)$. The weight of a walk in D is defined to be the product of the weights of all of the arcs on the walk. $A_{ij}^{(m)}$ is the sum of weights of all walks with length m from i to j (see [2]). We now have

Lemma 1: $a_{1j}^{(m)} = a_{jj}^{(m+1-j)}$.

Proof: Consider the sum of weights of all walks with length m from 1 to j $(j = 1, 2, 3, \ldots, n)$. For $1 \leq m \leq k-1$,

$$a_{1j}^{(m)} = \begin{cases} 1 & \text{if } m = j-1 \\ 0 & \text{otherwise.} \end{cases}$$

Clearly,

$$a_{jj}^{(m+1-j)} = \begin{cases} 1 & \text{if } m = j-1 \\ 0 & \text{if } j \leq m \leq k-1. \end{cases}$$

Now let $m > k-1$. The walks of length m from 1 to j must be of the form

$$1 \to 2 \to \cdots \to j \to \cdots \to k \to \cdots \to j.$$

Eliminating the path from 1 to j, we see that the preceding walks are in one-to-one correspondence with the walks of length $m - j + 1$ from j to j.

Since the weight of the path $1 \to 2 \to 3 \to \cdots \to j$ is 1, we have

$$a_{1j}^{(m)} = a_{jj}^{(m+1-j)}. \quad \blacksquare$$

Lemma 2: $a_{jj}^{(m)} = \sum_{i=1}^{j} \alpha_{k-i+1} f^{(m-k+i-1)}$ $(j = 1, 2, \ldots, k-1, k)$,

where

$$f^{(t)} = 0 \ (t < 0), \quad f^{(0)} = 1,$$

and
$$f^{(m)} = \sum_{\substack{s_1 + 2s_2 + \cdots + ks_k = m \\ s_i \geq 0 \ (i = 1, 2, \ldots, k)}} \binom{s_1 + s_2 + \cdots + s_k}{s_1, s_2, \ldots, s_k} \alpha_1^{s_1} \alpha_2^{s_2} \cdots \alpha_k^{s_k}$$

Proof: From the digraph D, it is not difficult to see that there are k classes of circuits from vertex k to k in D as given in the following table.

NAME	CIRCUIT	LENGTH	WEIGHT
C_1	$k \to k$	1	α_1
C_2	$k \to (k-1) \to k$	2	α_2
C_3	$k \to (k-2) \to (k-1) \to k$	3	α_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$
C_k	$k \to 1 \to 2 \to \cdots \to k$	k	α_k

Hence, any walk with length m from k to k must consist of s_1 C_1's, s_2 C_2's, ..., s_k C_k's.

The walks with length m from j to j, $1 \le j \le k-1$, have one of the j following forms:

NAME	CIRCUIT
Form (1)	$j \to \cdots \to k \to \cdots \to k \to 1 \to 2 \to \cdots \to j$ -----path----- ----------path---------- where $k \to \cdots \to k$ means passing through many circuits
Form (2)	$j \to \cdots \to k \to \cdots \to k \to 2 \to 3 \to \cdots \to j$
Form (3)	$j \to \cdots \to k \to \cdots \to k \to 3 \to 4 \to \cdots \to j$
$\vdots$	$\vdots$
Form (j)	$j \to \cdots \to k \to \cdots \to k \to j$

Clearly, the front path and the back path in form (i), where $i = 1, 2, \ldots, j$, together give a circuit C_{k-i+1}. Namely, there must be a circuit of length $k - i + 1$. Thus, for any fixed i ($1 \le i \le j$),

$$a_{jj}^{(m)} = \sum_{i=1}^{j} \sum_{\substack{s_1+2s_2+\cdots+ks_k=m \\ s_t \ge 0,\ t \ne k-i+1 \\ s_t \ge 1,\ t = k-i+1}} \binom{s_1+s_2+\cdots+(s_{k-i+1}-1)+\cdots+s_k}{s_1,\ s_2,\ \ldots,\ (s_{k-i+1}-1),\ \ldots,\ s_k} \alpha_1^{s_1}\alpha_2^{s_2}\cdots\alpha_k^{s_k}$$

$$= \sum_{i=1}^{j} \alpha_{k-i+1} \sum_{\substack{s_1+2s_2+\cdots+ks_k=m-k+i-1 \\ s_t \ge 0\ (t=1,\ldots,k)}} \binom{s_1+s_2+\cdots+s_k}{s_1,\ s_2,\ \ldots,\ s_k} \alpha_1^{s_1}\alpha_2^{s_2}\cdots\alpha_k^{s_k}, \quad 1 \le j \le k.$$

For convenience, let

$$f^{(m)} = f^{(m)}(\alpha_1, \alpha_2, \ldots, \alpha_k)$$

$$= \sum_{\substack{s_1+2s_2+\cdots+ks_k=m \\ s_t \ge 0\ (t=1,\ldots,k)}} \binom{s_1+s_2+\cdots+s_k}{s_1,\ s_2,\ \ldots,\ s_k} \alpha_1^{s_1}\alpha_2^{s_2}\cdots\alpha_k^{s_k}.$$

Hence,

$$a_{jj}^{(m)} = \sum_{i=1}^{j} \alpha_{k-i+1} f^{(m-k+i-1)}, \quad 1 \le j \le k. \quad \blacksquare$$

Lemma 3: For $f^{(m)}$, we have the following recurrence:

$$f^{(m)} = a_{kk}^{(m)} = \sum_{i=1}^{k} \alpha_{k-i+1} f^{(m-k+i-1)}.$$

Proof: According to the preceding analysis,

$$a_{kk}^{(m)} = \sum_{\substack{s_1 + 2s_2 + \cdots + ks_k = m \\ s \geq 0 \ (t = 1, \ldots, k)}} \binom{s_1 + s_2 + \cdots + s_k}{s_1, s_2, \ldots, s_k} \alpha_1^{s_1} \alpha_2^{s_2} \ldots \alpha_k^{s_k} = f^{(m)}.$$

By Lemma 2,

$$a_{kk}^{(m)} = \sum_{i=1}^{k} \alpha_{k-i+1} f^{(m-k+i-1)}.$$

Thus,

$$f^{(m)} = a_{kk}^{(m)} = \sum_{i=1}^{k} \alpha_{k-i+1} f^{(m-k+i-1)}. \quad \blacksquare$$

Theorem: The solution of the recurrence relation (1) is

$$u_m = \sum_{j=1}^{k} c_{j-1} \sum_{i=1}^{j} \alpha_{k-i+1} f^{(m-k-j+i)} + \sum_{j=1}^{m-k+1} b_{j-1} f^{(m+1-k-j)} \tag{9}$$

$$u_m = \sum_{j=1}^{k} c_{j-1} \sum_{i=1}^{j} \alpha_{k-i+1} \sum_{\substack{s_1 + 2s_2 + \cdots + ks_k = m-k+i-j \\ s_t \geq 0 \ (t = 1, \ldots, k)}} \binom{s_1 + s_2 + \cdots + s_k}{s_1, s_2, \ldots, s_k} \alpha_1^{s_1} \alpha_2^{s_2} \ldots \alpha_k^{s_k}$$

$$+ \sum_{j=1}^{m-k+1} b_{j-1} \sum_{\substack{s_1 + 2s_2 + \cdots + ks_k = m-k-j+1 \\ s_t \geq 0 \ (t = 1, \ldots, k)}} \binom{s_1 + s_2 + \cdots + s_k}{s_1, s_2, \ldots, s_k} \alpha_1^{s_1} \alpha_2^{s_2} \ldots \alpha_k^{s_k}.$$

Proof: By (7) and (8),

$$u_m = a^{(m)} = \sum_{j=1}^{k} c_{j-1} a_{1j}^{(m)} + \sum_{j=1}^{m-k+1} b_{j-1} a_{1k}^{(m-j)}$$

$$= \sum_{j=1}^{k} c_{j-1} a_{jj}^{(m+1-j)} + \sum_{j=1}^{m-k+1} b_{j-1} a_{kk}^{(m-k+1-j)} \quad \text{(Lemma 1)}$$

$$= \sum_{j=1}^{k} c_{j-1} \sum_{i=1}^{j} \alpha_{k-i+1} f^{(m-j-k+i)} + \sum_{j=1}^{m-k+1} b_{j-1} f^{(m-k+1-j)} \quad \text{(Lemmas 2 and 3).} \quad \blacksquare$$

Corollary 1:

$$u_m = \alpha_{k-1} f^{(m-k+1)} + \sum_{j=1}^{k-1} c_{j-1} \sum_{i=1}^{j} \alpha_{k-i+1} f^{(m-k-j+i)} + \sum_{j=1}^{m-k+1} b_{j-1} f^{(m+1-k-j)}.$$

Proof: This formula follows by using Lemma 3 and (9).

Corollary 2: The homogeneous recurrence (1) with constant coefficient has the solution

$$u_m = \alpha_{k-1} f^{(m-k+1)} + \sum_{j=1}^{k-1} c_{j-1} \sum_{i=1}^{j} \alpha_{k-i+1} f^{(m-k-j+i)}.$$

Corollary 3: The recurrence relation

$$\begin{cases} u_{n+k} = \alpha u_{n+r} + \beta u_n + b_n \\ u_0 = c_0, \ u_1 = c_1, \ \ldots, \ u_{k-1} = c_{k-1} \quad (1 \leq l \leq k-1) \end{cases} \tag{10}$$

has the solution

$$u_m = \sum_{j=0}^{r-1} c_j \beta f^{(m-k-j)} + \sum_{j=r}^{k-1} c_j f^{(m-j)} + \sum_{j=1}^{m-k+1} b_{j-1} f^{(m+1-k-j)},$$

where

$$f^{(m)} = \sum_{\substack{kx + (k-r)y = m \\ x, y \geq 0}} \binom{x+y}{y} \beta^x \alpha^y \quad (m \geq 0).$$

Proof: Let $\alpha_k = \beta$, $\alpha_{k-r} = \alpha$, and $\alpha_i = 0$, otherwise, in (1). By (9),

$$\begin{aligned} u_m &= \sum_{j=1}^{r} c_{j-1} \beta f^{(m-k-j+1)} + \sum_{j=r+1}^{k} c_{j-1}(\beta f^{(m-k-j+1)} + \alpha f^{(m-k-j+r+1)}) \\ &\quad + \sum_{j=1}^{m-k+1} b_{j-1} f^{(m+1-k-j)} \\ &= \sum_{j=0}^{r} c_{j-1} \beta f^{(m-k-j+1)} + \sum_{j=r+1}^{k} c_{j-1} f^{(m-j+1)} + \sum_{j=1}^{m-k+1} b_{j-1} f^{(m+1-k-j)} \qquad \text{(Lemma 3)} \\ &= \sum_{j=0}^{r-1} c_j \beta f^{(m-k-j)} + \sum_{j=r}^{k-1} c_j f^{(m-j)} + \sum_{j=1}^{m-k+1} b_{j-1} f^{(m+1-k-j)} \end{aligned}$$

where

$$f^{(m)} = \sum_{\substack{kx + (k-r)y = m \\ x, y \geq 0}} \binom{x+y}{y} \beta^x \alpha^y. \quad \blacksquare$$

When $b_n = 0$ in (10), Corollary 3 coincides with a result in [3]. When $b_n = 0$, $\alpha = \beta = 1$, $l = 1$, $k = 2$, and $c_0 = c_1 = 1$,

$$\begin{aligned} u_m &= c_0 f^{(m-2)} + c_1 f^{(m-1)} = f^{(m-2)} + f^{(m-1)} \\ &= f^{(m)} = \sum_{\substack{2x + y = m \\ x, y \geq 0}} \binom{x+y}{y} = \sum_{k=0}^{[m/2]} \binom{m-k}{k}, \end{aligned}$$

which is the combinatorial expression of the Fibonacci series.

Example 1: $F_{n+5} = 2F_{n+4} + 3F_n + (2n - 1)$

$F_0 = 1,\ F_1 = 0,\ F_2 = 1,\ F_3 = 2,\ F_4 = 3.$

Solution: $k = 5,\ l = 4,\ \alpha = 2,\ \beta = 3,\ b_n = 2n - 1$

$c_0 = 1,\ c_1 = 0,\ c_2 = 1,\ c_3 = 2,\ c_4 = 3.$

By Formula (10), one easily finds

$$\begin{aligned} F_n &= 3 \sum_{x=0}^{[(n-5)/5]} \binom{n-4x-5}{x} 3^x 2^{n-5x-5} + 3 \sum_{x=0}^{[(n-7)/5]} \binom{n-4x-7}{x} 3^x 2^{n-5x-7} \\ &\quad + 6 \sum_{x=0}^{[(n-8)/5]} \binom{n-4x-8}{x} 3^x 2^{n-5x-8} + 3 \sum_{x=0}^{[(n-4)/5]} \binom{n-4x-4}{x} 3^x 2^{n-5x-4} \\ &\quad + \sum_{j=1}^{n-4} (2j-3) \sum_{x=0}^{[(n-4-j)/5]} \binom{n-4x-4-j}{x} 3^x 2^{n-5x-4-j}. \end{aligned}$$

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