THE TETRANACCI SEQUENCE AND GENERALIZATIONS

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1. Introduction

Many papers concerning a variety of generalizations of the Fibonacci sequence have appeared, primarily in *The Fibonacci Quarterly*, in recent years. Horadam [1] was one of the first to initiate this interest when he changed the two initial terms of the Fibonacci sequence from 0, 1 to H_0, H_1, arbitrary integers, while maintaining the recurrence relation. He remarked in [1] that there are fundamentally two ways in which the Fibonacci sequence may be generalized; namely, either the recurrence relation can be changed or the initial terms can be altered. The two techniques can be combined, of course. Of the two alterations, a change in the recurrence relation seems to lead to greater complexity in the properties of the resulting sequence.

Some generalizations have been given names. The Tribonacci sequence, $\{T_n\}$, is defined by

(1) $$T_n = T_{n-1} + T_{n-2} + T_{n-3} \quad (n \geq 3), \quad T_0 = 0,\ T_1 = T_2 = 1.$$

A *generalized* Tribonacci sequence results when the recurrence relation is the same and T_0, T_1, T_2 are arbitrary. The Tribonacci sequence and this particular generalization have been examined rather extensively in the literature. See, for example, [2], [3], [4], [5], [6], [7].

The Tetranacci sequence, $\{M_n\}$, is defined by

(2) $$M_n = M_{n-1} + M_{n-2} + M_{n-3} + M_{n-4} \quad (n \geq 4), \quad M_0 = M_1 = 0,\ M_2 = M_3 = 1.$$

The first mention of the Tetranacci sequence seems to have occurred in [2], and it has received further brief attention or reference in [8], [9], [10], [11], [12]. Some writers have used the name "Quadranacci" (Latin) instead of "Tetranacci" (Greek). We use the latter, as in [2].

The characteristics and properties of the Tetranacci sequence apparently have not been examined in detail, and that, along with an examination of the generalization which occurs when the four initial terms are chosen as arbitrary integers, is the purpose of this paper.

As the recurrence relation and initial terms of Fibonacci-type sequences become more general, we quite naturally expect that the relationships among terms and the formal properties of the resulting sequences will become more complicated and complex, and this indeed is true. Nevertheless, by employing appropriate techniques, particularly by using vector and matrix methods, a number of properties of the Tetranacci sequence and generalizations and identities involving terms of these sequences are found and proved.

2. Fundamental Properties

As we begin an examination of the Tetranacci sequence and generalizations, two "companion" sequences emerge and are considered along with (2). These sequences are designated $\{N_n\}$ and $\{S_n\}$ and are defined as follows:

(3) $$N_n = N_{n-1} + N_{n-2} + N_{n-3} + N_{n-4} \quad (n \geq 4), \quad N_0 = N_2 = 0,\ N_1 = N_3 = 1,$$

(4) $$S_n = S_{n-1} + S_{n-2} + S_{n-3} + S_{n-4} \quad (n \geq 4), \quad S_0 = S_3 = 1,\ S_1 = S_2 = 0.$$

The sequences $\{N_n\}$ and $\{S_n\}$ have the same recurrence relation as $\{M_n\}$ but different initial terms. The initial terms are, in fact, two distinct permutations of the four initial terms of $\{M_n\}$. It can be shown also that these two companion sequences are further related to $\{M_n\}$ by

(5) $N_n = M_{n-1} + M_{n-2} + M_{n-3} \quad (n \geq 3),$

(6) $S_n = M_{n-1} + M_{n-2} \quad (n \geq 2).$

We define the *generalized* Tetranacci sequence, $\{\mu_n\}$, as

(7) $\mu_n = \mu_{n-1} + \mu_{n-2} + \mu_{n-3} + \mu_{n-4} \quad (n \geq 4)$

where μ_0, μ_1, μ_2, μ_3 are arbitrary integers.

The analogous *generalized* companion sequences, $\{\nu_n\}$ and $\{\sigma_n\}$, then become

(8) $\nu_n = \nu_{n-1} + \nu_{n-2} + \nu_{n-3} + \nu_{n-4} \quad (n \geq 4)$

or, alternately,

(9) $\nu_n = \mu_{n-1} + \mu_{n-2} + \mu_{n-3} \quad (n \geq 3),$

where $\nu_0 = \mu_1 - \mu_0$, $\nu_1 = \mu_2 - \mu_1$, $\nu_2 = \mu_3 - \mu_2$, $\nu_3 = \mu_2 + \mu_1 + \mu_0$,

and

(10) $\sigma_n = \sigma_{n-1} + \sigma_{n-2} + \sigma_{n-3} + \sigma_{n-4} \quad (n \geq 4)$

or, alternately,

(11) $\sigma_n = \mu_{n-1} + \mu_{n-2} \quad (n \geq 2),$

where $\sigma_0 = \mu_2 - \mu_1 - \mu_0$, $\sigma_1 = \mu_3 - \mu_2 - \mu_1$, $\sigma_2 = \mu_1 + \mu_0$, $\sigma_3 = \mu_2 + \mu_1$.

The choice of the initial terms of $\{\nu_n\}$ and $\{\sigma_n\}$ is not arbitrary but is determined by their relationship to $\{\mu_n\}$.

The table below gives values of the three sequences $\{M_n\}$, $\{N_n\}$, and $\{S_n\}$ for $n = 0$ to 18.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
M_n	0	0	1	1	2	4	8	15	29	56	108	208	401	773	1490	2872	5536	10,671	20,569
N_n	0	1	0	1	2	4	7	14	27	52	100	193	372	717	1382	2664	5135	9,898	19,079
S_n	1	0	0	1	2	3	6	12	23	44	85	164	316	609	1174	2263	4362	8,408	16,207

The analogue of Binet's formula for the Fibonacci sequence can be derived for $\{M_n\}$ and $\{\mu_n\}$. In [7] Spickerman and in [3] Waddill and Sacks derived the analogue of Binet's formula for the Tribonacci sequence and later in [8] Spickerman and Joyner generalized the result obtained in [7] to recursive sequences of order K. Since the Tetranacci sequence is a variation of the recursive sequence of order 4 in [8], the formula there may be adapted to give Binet's formula for the Tetranacci sequence; namely,

(12) $M_n = A_1 r_1^n + A_2 r_2^n + A_3 r_3^n + A_4 r_4^n,$

where A_i are constants and r_i are the four distinct roots of

$$x^4 - x^3 - x^2 - x - 1 = 0.$$

Binet's formula for μ_n is the same as (12) except that the A_i are functions of μ_0, μ_1, μ_2, μ_3. The A_i and r_i in (12) may be computed routinely but the resulting formula is long and cumbersome; hence, it is not written explicitly here nor used in the sequel.

A useful means of representing the recurrence relation of the Tetranacci sequence is by employing what we call the T-matrix, the analogue of the Q-matrix [13] which has been widely used in establishing properties of the Fibonacci sequence.

The T-matrix is defined to be

$$(13) \qquad T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Induction proofs may be used to establish

$$(14) \qquad \begin{bmatrix} M_n \\ M_{n-1} \\ M_{n-2} \\ M_{n-3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}^{n-3} \begin{bmatrix} M_3 \\ M_2 \\ M_1 \\ M_0 \end{bmatrix},$$

$$(15) \qquad \begin{bmatrix} \mu_n \\ \mu_{n-1} \\ \mu_{n-2} \\ \mu_{n-3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}^{n-3} \begin{bmatrix} \mu_3 \\ \mu_2 \\ \mu_1 \\ \mu_0 \end{bmatrix},$$

and

$$(16) \qquad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}^{n} = \begin{bmatrix} M_{n+2} & N_{n+2} & S_{n+2} & M_{n+1} \\ M_{n+1} & N_{n+1} & S_{n+1} & M_n \\ M_n & N_n & S_n & M_{n-1} \\ M_{n-1} & N_{n-1} & S_{n-1} & M_{n-2} \end{bmatrix}.$$

The right side of equation (16) indicates a reason for calling $\{N_n\}$ and $\{S_n\}$ "companion" sequences of $\{M_n\}$: both occur naturally in successive powers of the T-matrix.

Although up to this point, we have restricted the subscripts of the Tetranacci sequence and generalizations to being nonnegative, we may remove that restriction and define $\{M_n\}$, $\{N_n\}$, $\{S_n\}$ and their corresponding generalizations for all n.

By writing the difference equation (2) as

$$(17) \qquad M_n = M_{n+4} - M_{n+3} - M_{n+2} - M_{n+1},$$

and choosing $n < 0$, then $n + 4$, $n + 3$, $n + 2$, and $n + 1$ are all greater than n, which allows us to define M_n by the four terms immediately *following* it. That is,

$$M_{-1} = M_3 - M_2 - M_1 - M_0,$$
$$M_{-2} = M_2 - M_1 - M_0 - M_{-1},$$

and so on.

We may obtain another useful definition of $M_n, n < 0$, by using the T-matrix. We first write (14) as

$$(18) \qquad \begin{bmatrix} M_n \\ M_{n+1} \\ M_{n+2} \\ M_{n+3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}^{n} \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \end{bmatrix}.$$

Now, in (18), if we replace n by $-n$, we have, for $n > 0$,

(19) $$\begin{bmatrix} M_{-n} \\ M_{-n+1} \\ M_{-n+2} \\ M_{-n+3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}^{-n} \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}^{n} \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \end{bmatrix},$$

which defines M_n for $n < 0$; and this definition using the T-matrix is equivalent to (17).

The sequences $\{N_n\}$, $\{S_n\}$, $\{\mu_n\}$, $\{\nu_n\}$, $\{\sigma_n\}$ may be defined for $n < 0$ in like manner.

We now establish some interesting and useful identities. Using (15) and (16), we may write

(20) $$\begin{bmatrix} \mu_{n+p} \\ \mu_{n+p-1} \\ \mu_{n+p-2} \\ \mu_{n+p-3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}^{p} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}^{n-3} \begin{bmatrix} \mu_3 \\ \mu_2 \\ \mu_1 \\ \mu_0 \end{bmatrix}$$

$$= \begin{bmatrix} M_{p+2} & N_{p+2} & S_{p+2} & M_{p+1} \\ M_{p+1} & N_{p+1} & S_{p+1} & M_p \\ M_p & N_p & S_p & M_{p-1} \\ M_{p-1} & N_{p-1} & S_{p-1} & M_{p-2} \end{bmatrix} \begin{bmatrix} \mu_n \\ \mu_{n-1} \\ \mu_{n-2} \\ \mu_{n-3} \end{bmatrix},$$

From which we conclude that

(21) $$\mu_{n+p} = M_{p+2}\mu_n + N_{p+2}\mu_{n-1} + S_{p+2}\mu_{n-2} + M_{p+1}\mu_{n-3}$$

or

(22) $$\mu_{n+p} = M_{n+2}\mu_p + N_{n+2}\mu_{p-1} + S_{n+2}\mu_{p-2} + M_{n+1}\mu_{p-3}.$$

By replacing N_{p+2} and S_{p+2} using (5) and (6), regrouping and then employing (9) and (11), we find that (21) and (22) may be written

(23) $$\mu_{n+p} = M_{p+2}\mu_n + M_{p+1}\nu_n + M_p\sigma_n + M_{p-1}\mu_{n-1}$$

or

(24) $$\mu_{n+p} = M_{n+2}\mu_p + M_{n+1}\nu_p + M_n\sigma_p + M_{n-1}\mu_{p-1}.$$

As special cases of (21) and (23), respectively, when $p = 0$, we have

$$\mu_n = M_{n-1}\mu_3 + N_{n-1}\mu_2 + S_{n-1}\mu_1 + M_{n-2}\mu_0$$

or

$$\mu_n = M_{n-1}\mu_3 + M_{n-2}\nu_3 + M_{n-3}\sigma_3 + M_{n-4}\mu_2.$$

We next consider the sequence $\{R_n\}$ which is defined by

$$R_0 = M_1,\ R_1 = S_2,\ R_2 = N_2,\ R_3 = M_2$$

and

(25) $$\begin{bmatrix} R_{3n} \\ R_{3n-1} \\ R_{3n-2} \\ R_{3n-3} \end{bmatrix} = \begin{bmatrix} M_{n+1} \\ N_{n+1} \\ S_{n+1} \\ M_n \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} R_3 \\ R_2 \\ R_1 \\ R_0 \end{bmatrix}.$$

The generating matrix of $\{R_n\}$ is the transpose of the T-matrix, and the terms of $\{R_n\}$ are generated in groups of three rather than singularly as in (14). It is evident that the sequence $\{R_n\}$ is merely a meshing of the three sequences $\{M_n\}$, $\{N_n\}$, $\{S_n\}$, and, consequently, its terms are not as "spread out" as the terms of either of these sequences individually. This latter property become useful in establishing identities later on.

The generalized sequence for $\{R_n\}$ is designated $\{\rho_n\}$ and is defined as expected by

$$\rho_0 = \mu_1,\ \rho_1 = \sigma_2,\ \rho_2 = \nu_2,\ \rho_3 = \mu_2$$

and

$$(26)\quad \begin{bmatrix} \rho_{3n} \\ \rho_{3n-1} \\ \rho_{3n-2} \\ \rho_{3n-3} \end{bmatrix} = \begin{bmatrix} \mu_{n+1} \\ \nu_{n+1} \\ \sigma_{n+1} \\ \mu_n \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} \rho_3 \\ \rho_2 \\ \rho_1 \\ \rho_0 \end{bmatrix}$$

$$= \begin{bmatrix} M_{n+1} & M_n & M_{n-1} & M_{n-2} \\ N_{n+1} & N_n & N_{n-1} & N_{n-2} \\ S_{n+1} & S_n & S_{n-1} & S_{n-2} \\ M_n & M_{n-1} & M_{n-2} & M_{n-3} \end{bmatrix} \begin{bmatrix} \rho_3 \\ \rho_2 \\ \rho_1 \\ \rho_0 \end{bmatrix}.$$

Identities analogous to (21) and (23) may now be written for the sequences $\{\nu_n\}$ and $\{\sigma_n\}$. Using (26) and writing

$$(27)\quad \begin{bmatrix} \mu_{n+p} \\ \nu_{n+p} \\ \sigma_{n+p} \\ \mu_{n+p-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}^{p} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}^{n-3} \begin{bmatrix} \mu_3 \\ \nu_3 \\ \sigma_3 \\ \mu_2 \end{bmatrix}$$

$$= \begin{bmatrix} M_{p+2} & M_{p+1} & M_p & M_{p-1} \\ N_{p+2} & N_{p+1} & N_p & N_{p-1} \\ S_{p+2} & S_{p+1} & S_p & S_{p-1} \\ M_{p+1} & M_p & M_{p-1} & M_{p-2} \end{bmatrix} \begin{bmatrix} \mu_n \\ \nu_n \\ \sigma_n \\ \mu_{n-1} \end{bmatrix},$$

from (27) we conclude that

$$(28)\quad \nu_{n+p} = N_{p+2}\mu_n + N_{p+1}\nu_n + N_p\sigma_n + N_{p-1}\mu_{n-1},$$

$$(29)\quad \nu_{n+p} = N_{n+2}\mu_p + N_{n+1}\nu_p + N_n\sigma_p + N_{n-1}\mu_{p-1},$$

or by (20) replacing μ_i with ν_i, we have

$$(30)\quad \nu_{n+p} = M_{p+2}\nu_n + N_{p+2}\nu_{n-1} + S_{p+2}\nu_{n-2} + M_{p+1}\nu_{n-3},$$

$$(31)\quad \nu_{n+p} = M_{n+2}\nu_p + N_{n+2}\nu_{p-1} + S_{n+2}\nu_{p-2} + M_{n+1}\nu_{p-3}.$$

Similarly,

$$(32)\quad \sigma_{n+p} = S_{p+2}\mu_n + S_{p+1}\nu_n + S_p\sigma_n + S_{p-1}\mu_{n-1},$$

$$(33)\quad \sigma_{n+p} = S_{n+2}\mu_p + S_{n+1}\nu_p + S_n\sigma_p + S_{n-1}\mu_{p-1},$$

$$(34)\quad \sigma_{n+p} = M_{p+2}\sigma_n + N_{p+2}\sigma_{n-1} + S_{p+2}\sigma_{n-2} + M_{p+1}\sigma_{n-3},$$

$$(35)\quad \sigma_{n+p} = M_{n+2}\sigma_p + N_{n+2}\sigma_{p-1} + S_{n+2}\sigma_{p-2} + M_{n+1}\sigma_{p-3}.$$

We may further generalize (21) to read

$$(36)\quad \mu_{n+p} = M_{p+k+2}\mu_{n-k} + N_{p+k+2}\mu_{n-k-1} + S_{p+k+2}\mu_{n-k-2} + M_{p+k+1}\mu_{n-k-3},$$

where k is any integer. Since $\{\mu_n\}$ has been defined for all n, all terms in (36) are defined even if a chosen value of k produces negative subscripts. Also equations (22)-(24) and (28)-(35) can be written in this more general way.

In the vector on the left side of (15) the terms

$$\mu_n,\ \mu_{n-1},\ \mu_{n-2},\ \mu_{n-3}$$

are clearly adjacent terms of the sequence $\{\mu_n\}$. By using appropriate matrices we can write a vector in which the four terms are not adjacent but are "spread out" in a prescribed manner.

By (21) we have, for arbitrary integers p, q, and r,

$$(37)\qquad \begin{bmatrix} \mu_{n+p} \\ \mu_{n+q} \\ \mu_{n+r} \\ \mu_n \end{bmatrix} = \begin{bmatrix} M_{p+2} & N_{p+2} & S_{p+2} & M_{p+1} \\ M_{q+2} & N_{q+2} & S_{q+2} & M_{q+1} \\ M_{r+2} & N_{r+2} & S_{r+2} & M_{r+1} \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mu_n \\ \mu_{n-1} \\ \mu_{n-2} \\ \mu_{n-3} \end{bmatrix}.$$

Using (23), (28), and (32), we conclude that

$$(38)\qquad \begin{bmatrix} \mu_{n+p} \\ \nu_{n+q} \\ \sigma_{n+r} \\ \mu_n \end{bmatrix} = \begin{bmatrix} M_{p+2} & M_{p+1} & M_p & M_{p-1} \\ N_{q+2} & N_{q+1} & N_q & N_{q-1} \\ S_{r+2} & S_{r+1} & S_r & S_{r-1} \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mu_n \\ \nu_n \\ \sigma_n \\ \mu_{n-1} \end{bmatrix}.$$

Equations (37) and (38) will be used later on.

3. Linear Sums

A number of linear sum identities were discovered and proved. We give some of these and write them in terms of the generalized Tetranacci sequence, even though each has as a special case the corresponding identity for the Tetranacci sequence. All the listed identities may be proved by induction, but that method of proof gives no clue about their discovery. We give one proof to indicate how these identities, in general, were discovered.

We have

$$(39)\qquad \sum_{i=0}^{n} \mu_i = \frac{1}{3}[\mu_{n+2} + 2\mu_n + \mu_{n-1} + 2\mu_0 + \mu_1 - \mu_3],$$

$$(40)\qquad \sum_{i=0}^{n} \mu_{2i+1} = \frac{1}{3}[2\mu_{2n+2} + \mu_{2n} - \mu_{2n-1} - 2\mu_0 + 2\mu_1 - 3\mu_2 + \mu_3],$$

$$(41)\qquad \sum_{i=0}^{n} \mu_{2i} = \frac{1}{3}[2\mu_{2n+1} + \mu_{2n-1} - \mu_{2n-2} + 4\mu_0 - \mu_1 + 3\mu_2 - 2\mu_3],$$

$$(42)\qquad \sum_{i=0}^{n} \mu_{3i} = \frac{1}{9}[4\mu_{3n+1} + 3\mu_{3n} - \mu_{3n-1} + \mu_{3n-2} + 5\mu_0 - 5\mu_1 - 3\mu_2 + 2\mu_3],$$

$$(43)\qquad \sum_{i=0}^{n} \mu_{3i+1} = \frac{1}{9}[4\mu_{3n+2} + 3\mu_{3n+1} - \mu_{3n} + \mu_{3n-1} + 2\mu_0 + 7\mu_1 - 3\mu_2 - \mu_3],$$

$$(44)\qquad \sum_{i=0}^{n} \mu_{3i+2} = \frac{1}{9}[4\mu_{3n+3} + 3\mu_{3n+2} - \mu_{3n+1} + \mu_{3n} - \mu_0 + \mu_1 + 6\mu_2 - 4\mu_3],$$

$$(45)\qquad \sum_{i=1}^{n} \mu_{4i} = \sum_{i=0}^{4n-1} \mu_i = \frac{1}{3}[\mu_{4n+1} + 2\mu_{4n-1} + \mu_{4n-2} + 2\mu_0 + \mu_1 - \mu_3],$$

$$(46)\qquad \sum_{i=1}^{n} \mu_{4i+1} = \sum_{i=1}^{4n} \mu_i = \frac{1}{3}[\mu_{4n+2} + 2\mu_{4n} + \mu_{4n-1} - \mu_0 + \mu_1 - \mu_3],$$

$$(47)\qquad \sum_{i=1}^{n} \mu_{4i+2} = \sum_{i=2}^{4n+1} \mu_i = \frac{1}{3}[\mu_{4n+3} + 2\mu_{4n+1} + \mu_{4n} - \mu_0 - 2\mu_1 - \mu_3],$$

$$(48)\qquad \sum_{i=1}^{n} \mu_{4i+3} = \sum_{i=3}^{4n+2} \mu_i = \frac{1}{3}[\mu_{4n+4} + 2\mu_{4n+2} + \mu_{4n+1} - \mu_0 - 2\mu_1 - 3\mu_2 - \mu_3].$$

Proof of (39): We write the following obvious equations;

$$\begin{aligned} \mu_0 + \mu_1 + \mu_2 &= \mu_4 - \mu_3 \\ \mu_1 + \mu_2 + \mu_3 &= \mu_5 - \mu_4 \\ \mu_2 + \mu_3 + \mu_4 &= \mu_6 - \mu_5 \\ &\cdots\cdots \\ \mu_{n-1} + \mu_n + \mu_{n+1} &= \mu_{n+3} - \mu_{n+2} \\ \mu_n + \mu_{n+1} + \mu_{n+2} &= \mu_{n+4} - \mu_{n+3}. \end{aligned}$$

Now, adding these equations, we have

$$\sum_{i=0}^{n} \mu_i + \sum_{i=0}^{n} \mu_i + \mu_{n+1} - \mu_0 + \sum_{i=0}^{n} \mu_i + \mu_{n+1} + \mu_{n+2} - \mu_0 - \mu_1 = \mu_{n+4} - \mu_3,$$

or

$$3\sum_{i=0}^{n} \mu_i = \mu_{n+4} - 2\mu_{n+1} - \mu_{n+2} + 2\mu_0 + \mu_1 - \mu_3,$$

which may be reduced easily to (39) by using (7) and dividing both sides by 3.
The remaining identities, (40)-(48), are derived using similar techniques.

4. Quadratic, Cubic, and Quartic Identities

An application of the T-matrix is in deriving and proving the quadratic identity

(49) $$M_{n+1}^2 + M_n^2 + M_{n-1}^2 + 2M(M_{n-1} + M_{n-2}) = M_{2n}.$$

Proof of (49): By (16), we have

(50) $$T^{2n} = \begin{bmatrix} M_{2n+2} & N_{2n+2} & S_{2n+2} & M_{2n-1} \\ M_{2n+1} & N_{2n+1} & S_{2n+1} & M_{2n-2} \\ M_{2n} & N_{2n} & S_{2n} & M_{2n-3} \\ M_{2n-1} & N_{2n-1} & S_{2n-1} & M_{2n-4} \end{bmatrix} = \begin{bmatrix} M_{n+2} & N_{n+2} & S_{n+2} & M_{n+1} \\ M_{n+1} & N_{n+1} & S_{n+1} & M_n \\ M_n & N_n & S_n & M_{n-1} \\ M_{n-1} & N_{n-1} & S_{n-1} & M_{n-3} \end{bmatrix}^2.$$

Now we carry out the matrix multiplication on the right side of (50) and equate the elements in the third row, first column on both sides of (50) to obtain

$$M_nM_{n+2} + N_nM_{n+1} + S_nM_n + M_{n-1}^2 = M_{2n}$$

which is equivalent to (49).

By equating corresponding elements in the fourth row, first column of (50), we obtain

(51) $$M_{n+2}M_n - M_n^2 + M_nM_{n-3} + M_{n-1}^2 + 2M_{n-1}M_{n-2} = M_{2n-1}.$$

The generalized versions of (49) and (51) are, respectively,

(52) $$\begin{aligned} &\mu_{n+1}^2 + \mu_n^2 + \mu_{n-1}^2 + 2\mu_n(\mu_{n-1} + \mu_{n-2}) \\ &= \mu_3\mu_{2n-1} + \mu_2(\mu_{2n} - \mu_{2n-1}) + \mu_1(\mu_{2n-2} + \mu_{2n-3}) + \mu_0\mu_{2n-2} \end{aligned}$$

and

(53) $$\begin{aligned} &\mu_{n+2}\mu_n - \mu_n^2 + \mu_n\mu_{n-3} + \mu_{n-1}^2 + 2\mu_{n-1}\mu_{n-2} \\ &= \mu_3\mu_{2n-2} + \mu_2(\mu_{2n-2} - \mu_{2n-6}) + \mu_1(\mu_{2n-3} + \mu_{2n-4}) + \mu_0\mu_{2n-3}. \end{aligned}$$

In (52), if we let $\mu_0 = \mu_1 = 0$ and $\mu_2 = \mu_3 = 1$, we have

$$M_{n+1}^2 + M_n^2 + M_{n-1}^2 + 2M_n(M_{n-1} + M_{n-2}) = M_{2n},$$

which is (49). By letting $p = n - 1$, $\mu_0 = \mu_1 = 0$, $\mu_2 = \mu_3 = 1$, and replacing n by $n + 1$ is (21), we obtain (49) also. However, (21) is not readily obtainable from (52) nor is (52) obtainable from (21).

The same technique used in the proof of (49) may be used to find and prove cubic identities. In this case, we use the fact that for the T-matrix,

$$T^{3n-2} = T^{n-1}T^{n-1}T^{n}, \tag{54}$$

and again after expanding and equating appropriate corresponding terms on each side of (54), we obtain, for example,

$$M_{3n} = M_{n+2}(R_1 \cdot C_1) + M_{n+1}(R_1 \cdot C_2) + M_n(R_1 \cdot C_3) + M_{n-1}(R_1 \cdot C_4), \tag{55}$$

where R_1 is the first row of T^{n-1}, C_i is the i^{th} column of T^{n-1} and $\cdot$ is the usual dot product of two vectors. The right side of (55) is clearly a cubic which we do not expand completely because of its length.

The analogue of (55) for $\{\mu_n\}$ may be written in a manner similar to the way in which we wrote (52).

We may continue using the above technique to find quartic, quintic, and higher-ordered relations, but it is clear that one side (the side involving powers) of the equation becomes exceedingly long and complex.

One of the oldest and perhaps best known identities for the Fibonacci sequence is

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^{n+1}, \tag{56}$$

which was derived first by R. Simson [14]. In [3], the identity analogous to (56) was found for the Tribonacci sequence. We now pursue a like identity for the Tetranacci sequence. The simplest one may be obtained as in [3] by considering the determinants of both sides of (16) to obtain

$$\begin{vmatrix} M_{n+2} & M_{n+1} & M_n & M_{n-1} \\ M_{n+1} & M_n & M_{n-1} & M_{n-2} \\ M_n & M_{n-1} & M_{n-2} & M_{n-3} \\ M_{n-1} & M_{n-2} & M_{n-3} & M_{n-4} \end{vmatrix} = -\begin{vmatrix} M_{n+2} & M_{n-1} & M_n & M_{n+1} \\ M_{n+1} & M_{n-2} & M_{n-1} & M_n \\ M_n & M_{n-3} & M_{n-2} & M_{n-1} \\ M_{n-1} & M_{n-4} & M_{n-3} & M_{n-2} \end{vmatrix} \tag{57}$$

$$= -\begin{vmatrix} M_{n+2} & N_{n+2} & S_{n+2} & M_{n+1} \\ M_{n+1} & N_{n+1} & S_{n+1} & M \\ M_n & N_n & S_n & M_{n-1} \\ M_{n-1} & N_{n-1} & S_{n-1} & M_{n-2} \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix}^{n} = (-1)^{n+1}.$$

We shall not expand the left side of (57), but it is clearly a quartic consisting of 24 terms.

We now consider some generalizations of (57). First, we rewrite (57) for the sequence $\{\mu_n\}$ to obtain

$$\begin{vmatrix} \mu_{n+2} & \mu_{n+1} & \mu_n & \mu_{n-1} \\ \mu_{n+1} & \mu_n & \mu_{n-1} & \mu_{n-2} \\ \mu_n & \mu_{n-1} & \mu_{n-2} & \mu_{n-3} \\ \mu_{n-1} & \mu_{n-2} & \mu_{n-3} & \mu_{n-4} \end{vmatrix} = (-1)^n \begin{vmatrix} \mu_6 & \mu_5 & \mu_4 & \mu_3 \\ \mu_5 & \mu_4 & \mu_3 & \mu_2 \\ \mu_4 & \mu_3 & \mu_2 & \mu_1 \\ \mu_3 & \mu_2 & \mu_1 & \mu_0 \end{vmatrix}, \tag{58}$$

a quartic expression independent of n except for sign.

Proof of (58): By (15), we have the following matrix equation:

$$\begin{bmatrix} \mu_{n+2} & \mu_{n+1} & \mu_n & \mu_{n-1} \\ \mu_{n+1} & \mu_n & \mu_{n-1} & \mu_{n-2} \\ \mu_n & \mu_{n-1} & \mu_{n-2} & \mu_{n-3} \\ \mu_{n-1} & \mu_{n-2} & \mu_{n-3} & \mu_{n-4} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}^{n-4} \begin{bmatrix} \mu_6 & \mu_5 & \mu_4 & \mu_3 \\ \mu_5 & \mu_4 & \mu_3 & \mu_2 \\ \mu_4 & \mu_3 & \mu_2 & \mu_1 \\ \mu_3 & \mu_2 & \mu_1 & \mu_0 \end{bmatrix}. \tag{59}$$

Now, by taking determinants of both sides of (59), we have (58).

As a special case of (58), consider the sequence $\{\alpha_n\}$ where $\alpha_0 = \alpha_1 = 0$, $\alpha_2 = 1$, $\alpha_4 = \alpha$, arbitrary. The determinant on the right side of (58) then becomes

$$(60)\qquad \begin{vmatrix} 4(\alpha+1) & 2(\alpha+1) & \alpha+1 & \alpha \\ 2(\alpha+1) & (\alpha+1) & \alpha & 1 \\ (\alpha+1) & \alpha & 1 & 0 \\ \alpha & 1 & 0 & 0 \end{vmatrix},$$

which is a quartic polynomial in α. Consequently, an algebraic integer $\alpha = \beta$ exists, which makes the determinant (60) zero. Thus, for any n, the sequence $\{\alpha_n\}$ whose initial terms are 0, 0, 1, β, where β is chosen so as to make (60) equal 0, always results in

$$\begin{vmatrix} \alpha_{n+2} & \alpha_{n+1} & \alpha_n & \alpha_{n-1} \\ \alpha_{n+1} & \alpha_n & \alpha_{n-1} & \alpha_{n-2} \\ \alpha_n & \alpha_{n-1} & \alpha_{n-2} & \alpha_{n-3} \\ \alpha_{n-1} & \alpha_{n-2} & \alpha_{n-3} & \alpha_{n-4} \end{vmatrix} = 0.$$

To obtain a more general form of (58), we first observe that the quartics on the left side of (57) and (58) involve seven adjacent terms in the sequences $\{M_n\}$ and $\{\mu_n\}$, respectively. We use the technique in the proof of (58) along with (37) to show that the terms of the quartic may be "spread out," so to speak, and that the number of terms involved may be as great as 16. Specifically, we prove the following identity:

$$(61)\qquad \begin{vmatrix} \mu_{n+m+r} & \mu_{n+p+r} & \mu_{n+q+r} & \mu_{n+r} \\ \mu_{n+m+s} & \mu_{n+p+s} & \mu_{n+q+s} & \mu_{n+s} \\ \mu_{n+m+t} & \mu_{n+p+t} & \mu_{n+q+t} & \mu_{n+t} \\ \mu_{n+m} & \mu_{n+p} & \mu_{n+q} & \mu_n \end{vmatrix}$$

$$= (-1)^{n-1} \begin{vmatrix} M_{r+1} & M_r & M_{r-1} \\ M_{s+1} & M_s & M_{s-1} \\ M_{t+1} & M_t & M_{t-1} \end{vmatrix} \begin{vmatrix} \mu_{m+3} & \mu_{p+3} & \mu_{q+3} & \mu_3 \\ \mu_{m+2} & \mu_{p+2} & \mu_{q+2} & \mu_2 \\ \mu_{m+1} & \mu_{p+1} & \mu_{q+1} & \mu_1 \\ \mu_m & \mu_p & \mu_q & \mu_0 \end{vmatrix},$$

like (58) a quartic expression independent of n except for sign.

Proof of (61): By (37) and (20), we have the following matrix equation:

$$(62)\qquad \begin{bmatrix} \mu_{n+m+r} & \mu_{n+p+r} & \mu_{n+q+r} & \mu_{n+r} \\ \mu_{n+m+s} & \mu_{n+p+s} & \mu_{n+q+s} & \mu_{n+s} \\ \mu_{n+m+t} & \mu_{n+p+t} & \mu_{n+q+t} & \mu_{n+t} \\ \mu_{n+m} & \mu_{n+p} & \mu_{n+q} & \mu_n \end{bmatrix}$$

$$= \begin{bmatrix} M_{r+2} & N_{r+2} & S_{r+2} & M_{r+1} \\ M_{s+2} & N_{s+2} & S_{s+2} & M_{s+1} \\ M_{t+2} & N_{t+2} & S_{t+2} & M_{t+1} \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mu_{n+m} & \mu_{n+p} & \mu_{n+q} & \mu_n \\ \mu_{n+m-1} & \mu_{n+p-1} & \mu_{n+q-1} & \mu_{n-1} \\ \mu_{n+m-2} & \mu_{n+p-2} & \mu_{n+q-2} & \mu_{n-2} \\ \mu_{n+m-3} & \mu_{n+p-3} & \mu_{n+q-3} & \mu_{n-3} \end{bmatrix}$$

$$= \begin{bmatrix} M_{r+2} & N_{r+2} & S_{r+2} & M_{r+1} \\ M_{s+2} & N_{s+2} & S_{s+2} & M_{s+1} \\ M_{t+2} & N_{t+2} & S_{t+2} & M_{t+1} \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}^{n-3} \begin{bmatrix} \mu_{m+3} & \mu_{p+3} & \mu_{q+3} & \mu_3 \\ \mu_{m+2} & \mu_{p+2} & \mu_{q+2} & \mu_2 \\ \mu_{m+1} & \mu_{p+1} & \mu_{q+1} & \mu_1 \\ \mu_m & \mu_p & \mu_q & \mu_0 \end{bmatrix}.$$

We take determinants of both sides of (62) to obtain (61) since, by using (5) and (6) and well-known determinant properties, we can show that

$$\begin{vmatrix} M_{r+2} & N_{r+2} & S_{r+2} & M_{r+1} \\ M_{s+2} & N_{s+2} & S_{s+2} & M_{s+1} \\ M_{t+2} & N_{t+2} & S_{t+2} & M_{t+1} \\ 1 & 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} M_{r+1} & M_r & M_{r-1} \\ M_{s+1} & M_s & M_{s-1} \\ M_{t+1} & M_t & M_{t-1} \end{vmatrix}.$$

For the sequence $\{M_n\}$, (61) becomes.

$$(63)\qquad \begin{vmatrix} M_{n+m+r} & M_{n+p+r} & M_{n+q+r} & M_{n+r} \\ M_{n+m+s} & M_{n+p+s} & M_{n+q+s} & M_{n+s} \\ M_{n+m+t} & M_{n+p+t} & M_{n+q+t} & M_{n+t} \\ M_{n+m} & M_{n+p} & M_{n+q} & M_n \end{vmatrix} = (-1)^{n-1} \begin{vmatrix} M_{r+1} & M_r & M_{r-1} \\ M_{s+1} & M_s & M_{s-1} \\ M_{t+1} & M_t & M_{t-1} \end{vmatrix} \begin{vmatrix} M_{m+1} & M_m & M_{m-1} \\ M_{p+1} & M_p & M_{p-1} \\ M_{q+1} & M_q & M_{q-1} \end{vmatrix}.$$

Several special cases of (61) are worth mentioning. First, let $q = t$, $s = p = 2t$, $m = r = 3t$, n arbitrary, to obtain

$$(64)\qquad \begin{vmatrix} \mu_{n+6t} & \mu_{n+5t} & \mu_{n+4t} & \mu_{n+3t} \\ \mu_{n+5t} & \mu_{n+4t} & \mu_{n+3t} & \mu_{n+2t} \\ \mu_{n+4t} & \mu_{n+3t} & \mu_{n+2t} & \mu_{n+t} \\ \mu_{n+3t} & \mu_{n+2t} & \mu_{n+t} & \mu_n \end{vmatrix}$$

$$= (-1)^{n-1} \begin{vmatrix} M_{3t+1} & M_{2t+1} & M_{t+1} \\ M_{3t} & M_{2t} & M_t \\ M_{3t-1} & M_{2t-1} & M_{t-1} \end{vmatrix} \begin{vmatrix} \mu_{3t+3} & \mu_{2t+3} & \mu_{t+3} & \mu_3 \\ \mu_{3t+2} & \mu_{2t+2} & \mu_{t+2} & \mu_2 \\ \mu_{3t+1} & \mu_{2t+1} & \mu_{t+1} & \mu_1 \\ \mu_{3t} & \mu_{2t} & \mu_t & \mu_0 \end{vmatrix},$$

which displays an interesting symmetry.

Another special case of (61), which displays even greater symmetry, is obtained by letting $q = t = n$, $p = s = 2n$, $m = r = 3n$. We then have

$$(65)\qquad \begin{vmatrix} \mu_{7n} & \mu_{6n} & \mu_{5n} & \mu_{4n} \\ \mu_{6n} & \mu_{5n} & \mu_{4n} & \mu_{3n} \\ \mu_{5n} & \mu_{4n} & \mu_{3n} & \mu_{2n} \\ \mu_{4n} & \mu_{3n} & \mu_{2n} & \mu_n \end{vmatrix}$$

$$= (-1)^{n-1} \begin{vmatrix} M_{3n+1} & M_{2n+1} & M_{n+1} \\ M_{3n} & M_{2n} & M_n \\ M_{3n-1} & M_{2n-1} & M_{n-1} \end{vmatrix} \begin{vmatrix} \mu_{3n+3} & \mu_{2n+3} & \mu_{n+3} & \mu_3 \\ \mu_{3n+2} & \mu_{2n+2} & \mu_{n+2} & \mu_2 \\ \mu_{3n+1} & \mu_{2n+1} & \mu_{n+1} & \mu_1 \\ \mu_{3n} & \mu_{2n} & \mu_n & \mu_0 \end{vmatrix}.$$

Note how all terms in the determinant on the left of (65) are n units apart, whereas those on the right occur contiguously in groups of three or four, and the groups are $n - 3$ units apart.

5. Concluding remarks

Many number-theoretic properties for the Fibonacci sequence quite expectedly do not extend to the Tetranacci sequence. However, the following divisibility properties hold:

(66) $\quad M_{5n-1} \equiv M_{5n} \equiv M_{5n+1} \equiv 0 \pmod 2,$

(67) $\quad M_{5n-2} \equiv M_{5n+2} \equiv 1 \pmod 2,$

(68) $\quad M_{5n} \equiv M_{5n+1} \equiv 0 \pmod 4,$

(69) $\quad M_{5n-2} \equiv 1 \pmod 4.$

Proof of (66) and (67): We consider the sequence $\{M_n\}$ (mod 2) and display the results in the following table:

n	0	1	2	3	4	5	6	7	8	9
$M_n \pmod 2$	0	0	1	1	0	0	0	1	1	0

From the table, it is clear that $\{M_n\} \pmod 2$ starts to repeat after five terms and, since the pattern of zeros and ones will then continue to repeat in the same order, we have

$$M_4 \equiv M_{5n-1} \equiv 0 \pmod 2, \quad M_5 \equiv M_{5n} \equiv 0 \pmod 2, \quad M_6 \equiv M_{5n+1} \equiv 0 \pmod 2,$$
$$M_3 \equiv M_{5n-2} \equiv 1 \pmod 2, \quad M_2 \equiv M_{5n+2} \equiv 1 \pmod 2.$$

Since by (66), M_{5n-1}, M_{5n}, M_{5n+1} are even, it is clear that three arbitrary adjacent terms of the Tetranacci sequence may have greatest common divisor greater than one. However, we can show that the greatest common divisor of

$$M_n,\ M_{n+1},\ M_{n+2},\ M_{n+3},$$

any *four* consecutive terms of $\{M_n\}$, is one.

This paper, quite clearly, is not intended as an exhaustive treatment of properties of the Tetranacci sequence and generalizations. Some fundamental identities and sufficient other results and techniques for proving them are given to indicate the rich and remarkable nature of this sequence and generalizations.

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Announcement

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