ON THE r^{th}-ORDER NONHOMOGENEOUS RECURRENCE RELATION AND SOME GENERALIZED FIBONACCI SEQUENCES

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1. Introduction

Consider the nonhomogeneous recurrence relation

(1.1) $$G_n = G_{n-1} + G_{n-2} + \sum_{j=0}^{k} \alpha_j n^j$$

with

$$G_0 = 1;\ G_1 = 1.$$

In [1], Asveld expressed G_n in terms of Fibonacci numbers F_n and F_{n-1} and in the parameters $\alpha_0, \alpha_1, \ldots, \alpha_k$. He proved that

(1.2) $$G_n = (1 - G_0^{(p)})F_n + (-G_1^{(p)} + G_0^{(p)})F_{n-1} + G_n^{(p)},$$

where $G_n^{(p)}$ is a particular solution of (1.1).

In this paper, we generalize this result in two ways: First, we generalize Asveld's result by taking the second-order recurrence relation as

$$T_n = PT_{n-1} + QT_{n-2} + \sum_{j=0}^{k} \beta_j n^j$$

with

$$T_0 = a;\ T_1 = b.$$

Second, we prove similar results for the third-order and the r^{th}-order recurrence relations; cf. also [6].

In Section 2, we prove the results for the generalized second-order recurrence relation. In Section 3, we prove the theorem for the third-order recurrence relation. In Section 4, we mention the results for the r^{th}-order recurrence relation.

2. Generalized Second-Order Relation

Let the second-order nonhomogeneous recurrence relation be given by

(2.1) $$T_n = PT_{n-1} + QT_{n-2} + \sum_{j=0}^{k} \beta_j n^j$$

with

$$T_0 = a;\ T_1 = b.$$

Let the homogeneous relation corresponding to (2.1) be written as

(2.2) $$S_n = PS_{n-1} + QS_{n-2}$$

with the same initial conditions as for T_n, viz.,

$$S_0 = a;\ S_1 = b.$$

Whenever necessary, we denote the sequence S_n with the initial conditions $S_0 = a$, $S_1 = b$ as $S_n(a, b)$. It is well known that the solution of (2.2) is given by

(2.3) $$S_n(a, b) = \frac{1}{\alpha_2 - \alpha_1}[(aP - b)(\alpha_1^n - \alpha_2^n) - a(\alpha_1^{n+1} - \alpha_2^{n+1})]$$

where α_1 and α_2 are distinct roots of the characteristic equation of (2.2); see [5].

Note that

(2.4) $$\alpha_1 + \alpha_2 = P; \ \alpha_1\alpha_2 = -Q.$$

Also,

(2.5) $$S_n(1, 0) = \frac{1}{\alpha_2 - \alpha_1}[P(\alpha_1^n - \alpha_2^n) - (\alpha_1^{n+1} - \alpha_2^{n+1})],$$

(2.6) $$S_n(0, 1) = -\frac{1}{\alpha_2 - \alpha_1}[\alpha_1^n - \alpha_2^n],$$

and

(2.7) $$S_n(1, 1) = \frac{1}{\alpha_2 - \alpha_1}[(P - 1)(\alpha_1^n - \alpha_2^n) - (\alpha_1^{n+1} - \alpha_2^{n+1})].$$

Theorem 2.1: The solution of (2.1) is given by

$$T_n = S_n(a, b) - S_n(1, 0)T_0^{(p)} - S_n(0, 1)T_1^{(p)} + T_n^{(p)},$$

where $S_n(a, b)$, $S_n(1, 0)$, and $S_n(0, 1)$ are given by (2.3), (2.5), and (2.6), respectively, and $T_n^{(p)}$ is a particular solution of (2.1).

Proof: The solution of (2.1) is given by

$$T_n = T_n^{(h)} + T_n^{(p)},$$

where $T_n^{(h)}$ is the solution of (2.2) and $T_n^{(p)}$ is a particular solution of (2.1).

Now

(2.8) $$T_n = c_1\alpha_1^n + c_2\alpha_2^n + T_n^{(p)},$$

where

$$T_0 = a; \ T_1 = b.$$

Therefore,

(2.9) $$\begin{cases} c_1 + c_2 = a - T_0^{(p)}, \\ c_1\alpha_1 + c_2\alpha_2 = b - T_1^{(p)}. \end{cases}$$

Solving (2.9) simultaneously, we get

$$c_1 = \frac{(a - T_0^{(p)})\alpha_2 - b + T_1^{(p)}}{\alpha_2 - \alpha_1} = \frac{(a - T_0^{(p)})(P - \alpha_1) - b + T_1^{(p)}}{\alpha_2 - \alpha_1}.$$

Hence,

(2.10) $$c_1 = \frac{\alpha_1(T_0^{(p)} - a) + aP - b - PT_0^{(p)} + T_1^{(p)}}{\alpha_2 - \alpha_1}.$$

Similarly,

(2.11) $$c_2 = \frac{\alpha_2(-T_0^{(p)} + a) - aP + b + PT_0^{(p)} - T_1^{(p)}}{\alpha_2 - \alpha_1}.$$

Thus, by using (2.10) and (2.11) in (2.8), we have

$$\begin{aligned} T_n &= \frac{1}{\alpha_2 - \alpha_1}[(aP - b - PT_0^{(p)} + T_1^{(p)})(\alpha_1^n - \alpha_2^n) \\ &\qquad - (a - T_0^{(p)})(\alpha_1^{n+1} - \alpha_2^{n+1})] + T_n^{(p)} \\ &= \frac{1}{\alpha_2 - \alpha_1}\{[(aP - b)(\alpha_1^n - \alpha_2^n) - a(\alpha_1^{n+1} - \alpha_2^{n+1})] \\ &\qquad - [P(\alpha_1^n - \alpha_2^n) - (\alpha_1^{n+1} - \alpha_2^{n+1})]T_0^{(p)} \\ &\qquad - [-(\alpha_1^n - \alpha_2^n)]T_1^{(p)} + T_n^{(p)}. \end{aligned}$$

By using (2.3), (2.5), and (2.6) we finally obtain

(2.12) $$T_n = S_n(a, b) - S_n(1, 0)T_0^{(p)} - S_n(0, 1)T_1^{(p)} + T_n^{(p)}.$$

Remarks:

(1) Note that, if $a = 1$, $b = 1$, $P = 1$, $Q = 1$, (2.12) reduces to Asveld's result given by (1.2). Here we use the fact that

$$S_n(1, 0) = -F_{n-1} + F_n = F_{n-2}, \quad S_n(0, 1) = F_{n-1}, \quad S_n(1,1) = F_n.$$

(2) To get a complete solution of (2.1), let the particular solution $T_n^{(p)}$ be given by

$$T_n^{(p)} = \sum_{i=0}^{k} A_i n^i.$$

Then, from (2.1) we get

$$\sum_{i=0}^{k} A_i n^i - P\sum_{i=0}^{k} A_i(n-1)^i - Q\sum_{i=0}^{k} A_i(n-2)^i - \sum_{i=0}^{k} \beta_i n^i = 0$$

or

$$\sum_{i=0}^{k} A_i n^i - \sum_{i=0}^{k}\left(\sum_{\ell=0}^{i} A_i\binom{i}{\ell}(-1)^{i-\ell}(P + Q2^{i-\ell})n^\ell\right) - \sum_{i=0}^{k} \beta_i n^i = 0.$$

For each i $(0 \le i \le k)$, we have

(2.13) $$A_i - \sum_{m=i}^{k} \gamma_{im} A_m - \beta_i = 0$$

where, for $m \ge i$,

$$\gamma_{im} = \binom{m}{i}(-1)^{m-i}(P + Q2^{m-i}).$$

From the recurrence relation (2.13), $A_k, \ldots, A_0$ can be computed where A_i is a linear combination of $\beta_i, \ldots, \beta_k$. To get a more explicit solution as in Asveld [1], we put

$$A_i = -\sum_{j=i}^{k} a_{ij}\beta_j,$$

where a_{ij} are as defined below. Then we get the following solution for (2.12):

$$T_n = S_n(a, b) + S_n(1, 0)\lambda_k^0 + S_n(0, 1)\lambda_k^1 - \sum_{j=0}^{k} \beta_j r_j(n),$$

where

$$\lambda^0 = \sum_{j=0}^{k} \beta_j a_{0j}, \quad \lambda_k^1 = \sum_{j=0}^{k} \beta_j \sum_{i=0}^{j} a_{ij}, \quad \text{and} \quad r_j(n) = \sum_{i=0}^{j} a_{ij} n^i.$$

Note that

$$\gamma_{ii} = P + Q, \quad a_{ii} = \frac{1}{P + Q - 1}, \quad \text{and} \quad a_{ij} = -\sum_{m=i+1}^{j} \gamma_{im} a_{mj}, \quad j > i.$$

(3) If $a = 2$, $b = 1$, $P = 1$, and $Q = 1$, the sequence $S_n(a, b)$ reduces to the Lucas sequence L_n. Then (2.12) reduces to

$$T_n = L_n - T_0^{(p)}F_n + (T_0^{(p)} - T_1^{(p)})F_{n-1} + T_n^{(p)}.$$

(4) We are grateful to the referee for pointing out references [6], [7], and [8]. It should be noted that our results are more general than those in [6]. One can also prove results similar to those in [6] and [7] without much difficulty.

3. Third-Order Recurrence Relation

Let the third-order recurrence relation be given by

(3.1) $$T_n = P_1 T_{n-1} + P_2 T_{n-2} + P_3 T_{n-3} + \sum_{j=0}^{k} \beta_j n^j.$$

Let the homogeneous relation corresponding to (3.1) be written as

(3.2) $$S_n = P_1 S_{n-1} + P_2 S_{n-2} + P_3 S_{n-3}.$$

Denote the sequence S_n by S_n^1, S_n^2, S_n^3, when

(3.3) $S_0 = 0,\ S_1 = 1,\ S_2 = P_1,$

(3.4) $S_0 = 1,\ S_1 = 0,\ S_2 = P_2,$ and

(3.5) $S_0 = 0,\ S_1 = 0,\ S_2 = P_3,$

respectively.

Denote the sequence T_n with initial conditions the same as (3.3), (3.4), and (3.5) by T_n^1, T_n^2, T_n^3, respectively. If α_1, α_2, α_3 are distinct roots of the characteristic equation corresponding to (3.2), then

$$S_n = c_1\alpha_1^n + c_2\alpha_2^n + c_3\alpha_3^n$$

with

(3.6) $$\alpha_1 + \alpha_2 + \alpha_3 = P_1; \quad \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3 = -P_2; \quad \alpha_1\alpha_2\alpha_3 = P_3.$$

Using standard methods, we obtain

$$S_n^1 = \frac{1}{\Delta}[\alpha_1^{n+1}(\alpha_3 - \alpha_2) - \alpha_2^{n+1}(\alpha_3 - \alpha_1) + \alpha_3^{n+1}(\alpha_2 - \alpha_1)],$$

$$S_n^2 = \frac{1}{\Delta}[\alpha_1^{n+1}(\alpha_3^2 - \alpha_2^2) - \alpha_2^{n+1}(\alpha_3^2 - \alpha_1^2) + \alpha_3^{n+1}(\alpha_2^2 - \alpha_1^2)],$$

$$S_n^3 = \frac{P_3}{\Delta}[\alpha_1^n(\alpha_3 - \alpha_2) - \alpha_2^n(\alpha_3 - \alpha_1) + \alpha_3^n(\alpha_2 - \alpha_1)],$$

where $$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \end{vmatrix} = (\alpha_3 - \alpha_2)(\alpha_3 - \alpha_1)(\alpha_2 - \alpha_1);$$ see [4].

By making use of (3.6), we easily get

$$S_n^2 = -P_1 S_n^1 + S_{n+1}^1, \quad S_n^3 = P_3 S_{n-1}^1.$$

For the sake of convenience, let T_n^1 be denoted by T_n in what follows.

Theorem 3.1: T_n is given in terms of S_n^1 by

$$T_n = -P_3 T_0^{(p)} S_{n-2}^1 + (P_1 T_1^{(p)} - T_2^{(p)}) S_{n-1}^1 + (1 - T_1^{(p)}) S_n^1 + T_n^{(p)}.$$

Proof: Let $T_n^{(h)}$ be the solution of (3.2) and $T_n^{(p)}$ be a particular solution of (3.1). Then

(3.7) $$T_n = T_n^{(h)} + T_n^{(p)}$$

where

(3.8) $$T_n^{(h)} = c_1\alpha_1^n + c_2\alpha_2^n + c_3\alpha_3^n$$

with initial conditions

$$T_0 = 0,\ T_1 = 1,\ T_2 = P_1.$$

Using these initial conditions, we have

$$\begin{aligned} c_1 + c_2 + c_3 &= -T_0^{(p)}, \\ c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 &= 1 - T_1^{(p)}, \\ c_1\alpha_1^2 + c_2\alpha_2^2 + c_3\alpha_3^2 &= P_1 - T_2^{(p)}. \end{aligned}$$

Solving these equations simultaneously, we get

$$\begin{aligned} c_1 &= \frac{\alpha_3 - \alpha_2}{\Delta}[-T_0^{(p)}\alpha_2\alpha_3 - (1 - T_1^{(p)})(\alpha_2 + \alpha_3) + (P_1 - T_2^{(p)})] \\ &= \frac{\alpha_3 - \alpha_2}{\Delta}\left[-\frac{P_3}{\alpha_1} T_0^{(p)} - (1 - T_1^{(p)})(P_1 - \alpha_1) + P_1 - T_2^{(p)}\right]. \end{aligned}$$

Similarly,

$$c_2 = -\frac{\alpha_3 - \alpha_1}{\Delta}\left[-\frac{P_3}{\alpha_2} T_0^{(p)} - (1 - T_1^{(p)})(P_1 - \alpha_2) + P_1 - T_2^{(p)}\right]$$

and

$$c_3 = \frac{\alpha_2 - \alpha_1}{\Delta}\left[-\frac{P_3}{\alpha_3} T_0^{(p)} - (1 - T_1^{(p)})(P_1 - \alpha_3) + P_1 - T_2^{(p)}\right].$$

Hence, substituting for c_1, c_2, c_3 in (3.8) and simplifying we get

$$\begin{aligned} T_n^{(h)} = \{&-P_3T_0^{(p)}[\alpha_1^{n-1}(\alpha_3 - \alpha_2) - \alpha_2^{n-1}(\alpha_3 - \alpha_1) + \alpha_3^{n-1}(\alpha_2 - \alpha_1)] \\ &-P_1 (1 - T_1^{(p)})[\alpha_1^n(\alpha_3 - \alpha_2) - \alpha_2^n(\alpha_3 - \alpha_1) + \alpha_3^n(\alpha_2 - \alpha_1)] \\ &+ (1 - T_1^{(p)})[\alpha_1^{n+1}(\alpha_3 - \alpha_2) - \alpha_2^{n+1}(\alpha_3 - \alpha_1) + \alpha_3^{n+1}(\alpha_2 - \alpha_1)] \\ &+ (P_1 - T_2^{(p)})[\alpha_1^n(\alpha_3 - \alpha_2) - \alpha_2^n(\alpha_3 - \alpha_1) + \alpha_3^n(\alpha_2 - \alpha_1)]\}/\Delta \\ = &-P_3T_0^{(p)}S_{n-2}^1 - P_1(1 - T_1^{(p)})S_{n-1}^1 + (2 - T_1^{(p)})S_n^1 + (P_1 - T_2^{(p)})S_{n-1}^1. \end{aligned}$$

On further simplification, (3.7) reduces to

(3.9) $$T_n = -P_3T_0^{(p)}S_{n-2}^1 + (P_1T_1^{(p)} - T_2^{(p)})S_{n-1}^1 + (1 - T_1^{(p)})S_n^1 + T_n^{(p)},$$

which is the required result.

Remarks:

(1) If $P_1 = 1$, $P_2 = 1$, $P_3 = 0$, and $T_0 = 0$, $T_1 = 1$, (3.1) and (3.2) reduce to the second-order relations (2.1) and (2.2) with $P = Q = 1$ and $a = 0$, $b = 1$. With the above values of P_1, P_2, and P_3, T_n given by (3.9) reduces to

$$T_n = (T_1^{(p)} - T_2^{(p)})S_{n-1}^1 + (1 - T_1^{(p)})S_n^1 + T_n^{(p)}.$$

We verify whether this equation reduces to (2.13) with $a = 0$, $b = 1$. Now

$$T_1^{(p)} - T_2^{(p)} = T_1 - T_1^{(h)} - T_2 + T_2^{(h)},$$

since $T_n = T_n^{(h)} + T_n^{(p)}$. Also,

$$T_1 = T_2 = 1 \quad \text{and} \quad T_2^{(h)} = T_1^{(h)} + T_0^{(h)}.$$

Therefore,

$$T_1^{(p)} - T_2^{(p)} = -T_1^{(h)} + T_2^{(h)} = T_0^{(h)} = T_0 - T_0^{(p)} = -T_0^{(p)},$$

since $T_0 = 0$. Thus,

(3.10) $$T_n = -T_0^{(p)}S_{n-1}^1 + (1 - T_1^{(p)})S_n^1 + T_n^{(p)}.$$

Note that here $S_n^1 = S_n(0, 1)$. Now

$$S_n(1, 0) = S_{n-1}(0, 1).$$

Hence, (3.10) reduces to

$$T_n = -S_n(1, 0)T_0^{(p)} + (1 - T_1^{(p)})S_n(0, 1) + T_n^{(p)},$$

which is identical with (2.12).

(2) On similar lines, we can prove the following:

$$T_n^2 = P_3(1 - T_0^{(p)})S_{n-2}^1 + (P_1T_1^{(p)} + P_2 - T_2^{(p)})S_{n-1}^1 - T_1^{(p)}S_n^1 + T_n^{(p)};$$
$$T_n^3 = -P_3T_0^{(p)}S_{n-2}^1 + (P_1T_1^{(p)} + P_3 - T_2^{(p)})S_{n-1}^1 - T_1^{(p)}S_n^1 + T_n^{(p)}.$$

(3) As in Remark (2) of Section 2, taking

$$T_n^{(p)} = \sum_{i=0}^{k} A_i n^i \quad \text{and} \quad A_i = -\sum_{j=i}^{k} a_{ij}\beta_j,$$

a_{ij} as defined below, the sequences T_n^1 can be expressed as

$$T_n^1 = P_3\lambda_k^0 S_{n-2}^1 + (-P_1\lambda_k^1 + \lambda_k^2)S_{n-1}^1 + (1 + \lambda_k^1)S_n^1 - \sum_{j=0}^{k}\beta_j r_j(n),$$

where

$$\lambda_k^0 = \sum_{j=0}^{k}\beta_j a_{0j}, \quad \lambda_k^1 = \sum_{j=0}^{k}\beta_j\sum_{i=0}^{j}a_{ij}, \quad \lambda_k^2 = \sum_{j=0}^{k}\beta_j\sum_{i=0}^{j}2^i a_{ij},$$

$$r_j(n) = \sum_{i=0}^{j}a_{ij}n^i, \quad a_{ij} = -\sum_{m=i+1}^{j}\delta_{im}a_{mj}, \; j > i,$$

and

$$\delta_{im} = \binom{m}{i}(-1)^{m-i}[P_1 + P_2 2^{m-i} + P_3 3^{m-i}].$$

(4) Similar results as above can be obtained for T_n^2 and T_n^3.

4. The r^{th}-Order Recurrence Relation

Let

(4.1) $$T_n = P_1T_{n-1} + P_2T_{n-2} + \cdots + P_{r-1}T_{n-r+1} + P_rT_{n-r} + \sum_{j=0}^{k}\beta_j n^j, \; r \ge 3,$$

be the r^{th}-order recurrence relation with three sets of initial conditions as

(4.2) $T_m = 0$, for $0 \le m \le r - 3$, $T_{r-2} = 1$, $T_{r-1} = P_1$,

(4.3) $T_m = 0$, for $0 < m < r - 1$, $T_0 = 1$, $T_{r-1} = P_2$,

(4.4) $T_m = 0$, for $0 \le m \le r - 2$, $T_{r-1} = P_3$.

The homogeneous part of (4.1) is the generalized r^{th}-order Fibonacci sequence. Let it be denoted by S_n so that

$$S_n = P_1S_{n-1} + P_2S_{n-2} + \cdots + P_rS_{n-r}.$$

We take the same initial conditions as in (4.2)-(4.4). Following the same method as in Section 3, we can prove the following results:

$$\begin{aligned} T_n^1 = {} & -P_rT_0^{(p)}S_{n-2}^1 + (P_{r-2}T_1^{(p)} + \cdots + P_1T_{r-2}^{(p)} - T_{r-1}^{(p)})S_{n-1}^1 \\ & + (1 + P_{r-3}T_1^{(p)} + \cdots + P_1T_{r-3}^{(p)} - T_{r-2}^{(p)})S_n^1 + \cdots \\ & + (P_1T_1^{(p)} - T_2^{(p)})S_{n+r-4}^1 - T_1^{(p)}S_{n+r-3}^1 + T_n^{(p)}; \end{aligned}$$

$$\begin{aligned} T_n^2 = {} & P_r(1 - T_0^{(p)})S_{n-2}^1 + (P_{r-2}T_1^{(p)} + \cdots + P_1T_{r-2}^{(p)} + P_2 - T_{r-1}^{(p)})S_{n-1}^1 \\ & + (P_{r-3}T_1^{(p)} + \cdots + P_1T_{r-3}^{(p)} - T_{r-2}^{(p)})S_n^1 + \cdots \\ & + (P_1T_1^{(p)} - T_2^{(p)})S_{n+r-4}^1 - T_1^{(p)}S_{n+r-3}^1 + T_n^{(p)}; \end{aligned}$$

and

$$T_n^3 = -P_r T_0^{(p)} S_{n-2}^1 + (P_{r-2} T_1^{(p)} + \cdots + P_1 T_{r-2}^{(p)} + P_3 - T_{r-1}^{(p)}) S_{n-1}^1$$
$$+ (P_{r-3} T_1^{(p)} + \cdots + P_1 T_{r-3}^{(p)} - T_{r-2}^{(p)}) S_n^1 + \cdots$$
$$+ (P_1 T_1^{(p)} - T_2^{(p)}) S_{n+r-4}^1 - T_1^{(p)} S_{n+r-3}^1 + T_n^{(p)}.$$

Here we denote S_n with initial conditions (4.2) by S_n^1 and T_n with initial conditions (4.2), (4.3), (4.4) by T_n^1, T_n^2, T_n^3, respectively.

Remarks:

(1) For $r = 3$, T_n^1 reduces to the result of Theorem 3.1.

(2) As in Remark (2) of Section 2, taking

$$T_n^{(p)} = \sum_{i=0}^{k} A_i n^i \quad \text{and} \quad A_i = -\sum_{j=i}^{k} a_{ij} \beta_j,$$

the sequence T_n^1 can be expressed as follows:

$$T_n^1 = P_r \lambda_k^0 S_{n-2}^1 - (P_{r-2} \lambda_k^1 + \cdots + P_1 \lambda_k^{r-2} - \lambda_k^{r-1}) S_{n-1}^1$$
$$+ (1 - P_{r-3} \lambda_k^1 - \cdots - P_1 \lambda_k^{r-3} + \lambda_k^{r-2}) S_n^1 + \cdots$$
$$+ (-P_1 \lambda_k^1 + \lambda_k^2) S_{n+r-4}^1 + \lambda^1 S_{n+r-3}^1 - \sum_{j=0}^{k} \beta_j r_j(n),$$

where

$$\lambda_0^k = \sum_{j=0}^{k} \beta_j a_{0j}, \qquad \lambda_k^1 = \sum_{j=0}^{k} \beta_j \sum_{i=0}^{j} a_{ij} \ell^i, \quad \ell = 1, 2, \ldots, r-1;$$

$$r_j(n) = \sum_{i=0}^{j} a_{ij} n^i, \quad a_{ij} = -\sum_{m=i+1}^{j} \delta_{im} a_{mj}, \ j > i;$$

and

$$\delta_{im} = \binom{m}{i} (-1)^{m-i} [P_1 + P_2 2^{m-i} + \cdots + P_r r^{m-i}].$$

(3) Similarly, we can write the values of T_n^2 and T_n^3.

(4) In [3], Asveld derived expressions for the family of differential equations corresponding to (1.1).

It is natural to ask whether such results can be proved for the r^{th}-order recurrence relation. This is the subject of our next paper.

References

1. P. R. J. Asveld. "A Family of Fibonacci-Like Sequences." *Fibonacci Quarterly 25.1* (1987):81-83.
2. P. R. J. Asveld. "Another Family of Fibonacci-Like Sequences." *Fibonacci Quarterly 25.4* (1987):361-64.
3. P. R. J. Asveld. "Fibonacci-Like Differential Equation with a Polynomial Nonhomogeneous Part." *Fibonacci Quarterly 27.2* (1989):303-09.
4. W. L. Ferrar. *Text Book of Determinants, Matrices and Algebraic Forms*. Oxford: Oxford University Press, 1941.
5. A. F. Horadam. "Basic Properties of Certain Generalized Sequences of Numbers." *Fibonacci Quarterly 3.3* (1965):161-76.
6. A. F. Horadam & A. G. Shannon. "Asveld's Polynomials $p_j(n)$." *Proceedings of the Third International Conference on Fibonacci Numbers and Their Applications*. Dordrecht: Kluwer, 1988.
7. A. F. Horadam. "Falling Factorial Polynomials of Generalized Fibonacci Type." *Proceedings of the Fourth International Conference on Fibonacci Numbers and Their Applications*. Dordrecht: Kluwer, 1990.
8. J. C. Turner. "Note on a Family of Fibonacci-Like Sequences." *Fibonacci Quarterly 27.3* (1989):229-32.