ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions.

Dedication. This year's column is dedicated to **Dr. A. P. Hillman** in recognition of his 27 years of devoted service as editor of the Elementary Problems Section. Devotees of this column are invited to thank Abe by dedicating their next proposed problem to Dr. Hillman.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-718 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove that $[(F_n + L_n)\alpha + (F_{n-1} + L_{n-1})]/2$ is a power of the golden ratio, α.

B-719 *Proposed by Herta T. Freitag, Roanoke, VA*

Dedicated to Dr. A. P. Hillman

Let P_n be the n^{th} Pell number (defined by $P_0 = 0$, $P_1 = 1$, and $P_{n+2} = 2P_{n+1} + P_n$ for $n \ge 0$). Let a be an odd integer. Show how to factor $P_{n+a}^2 + P_n^2$ into a product of Pell numbers.

How should this problem be modified if a is even?

B-720 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

Dedicated to Dr. A. P. Hillman

Find a closed form expression for

$$S_n = \sum_{h+k=2n} F_h F_k$$

where the sum is taken over all pairs of positive integers (h, k) such that $h + k = 2n$ and $h \le k$.

B-721 *Proposed by Russell Jay Hendel, Dowling College, Oakdale, NY*

Dedicated to Dr. A. P. Hillman

Brittany is going to ascend an m step staircase. At any time she is just as likely to stride up one step as two steps. For a positive integer k, find the probability that she ascends the whole staircase in k strides.

B-722 *Proposed by H.-J. Seiffert, Berlin, Germany*

Dedicated to Dr. A. P. Hillman

Define the Fibonacci polynomials by

$$F_0(x) = 0,\ F_1(x) = 1,\ F_n(x) = xF_{n-1}(x) + F_{n-2}(x),\ \text{for } n \geq 2.$$

Show that for all nonnegative integers n,

$$\int_0^\infty \frac{dx}{(x^2+1)F_{2n+1}(2x)} = \frac{\pi}{4n+2}.$$

B-723 *Proposed by Bruce Dearden & Jerry Metzger, University of North Dakota, Grand Forks, ND*

(a) Show that for $n \equiv 2 \pmod 4$,

$$F_{n+1}(F_n^2 + F_n - 1) \text{ divides } F_n^n(F_n^2 + F_{n+1}) - 1.$$

(b) What is the analog of (a) for $n \equiv 0 \pmod 4$?

SOLUTIONS

A Congruence for L_{2^n}

B-694 *Proposed by Sahib Singh, Clarion U. of Pennsylvania, Clarion, PA*

Prove that $L_{2^n} \equiv 7 \pmod{40}$ for $n \geq 2$.

Solution 1 by Lawrence Somer, Washington, DC

It is well known (see, for example, formula 17c of [1]) that

$$L_{2m} = L_m^2 - 2(-1)^m.$$

Letting $m = 2^{n-1}$ gives

$$L_{2^n} = L_{2^{n-1}}^2 - 2$$

for $n \geq 2$. For the case $n = 2$, we have

$$L_{2^2} = L_4 = 7.$$

Suppose that $L_{2^n} \equiv 7 \pmod{40}$ for some $n \geq 2$. Then

$$L_{2^{n+1}} = L_{2^n}^2 - 2 \equiv 7^2 - 2 \equiv 7 \pmod{40}$$

and the result is true for $n + 1$.

The result now follows for all $n \geq 2$ by mathematical induction.

Reference

1. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section*. Ellis Harwood Ltd., West Sussex, England, 1989.

Solution 2 by Russell Jay Hendel (paraphrased), Dowling College, Oakdale, NY

Writing down the Lucas sequence modulo 40, we obtain

$$2, 1, 3, 4, 7, 11, 18, 29, 7, 36, 3, 39, 2, 1, \ldots$$

and we thus see that the sequence repeats every 12 terms. That is,

$$L_a \equiv L_b \pmod{40} \quad \text{if } a \equiv b \pmod{12}.$$

Modulo 12, the sequence 2^n for $n \geq 2$ proceeds 4, 8, 4, 8, 4, 8, ..., so

$$2^n \equiv 4 \text{ or } 8 \pmod{12} \quad \text{for } n \geq 2.$$

Thus,

$$L_{2^n} \equiv L_4 \text{ or } L_8 \pmod{40} \quad \text{for } n \geq 2.$$

But $L_4 = 7$ and $L_8 = 47$ are both congruent to 7 modulo 40. Hence,

$$L_{2^n} \equiv 7 \pmod{40} \quad \text{for } n \geq 2.$$

None of the solvers submitted any generalizations to this problem. The problem cries out for a Fibonacci analog. Many such are possible, for example:

$$F_{2^n} \equiv 21 \pmod{42} \quad \text{for } n \geq 3.$$

The editor will normally be pleased to publish any related results or generalizations readers find for problems published in this section.

Also solved by Charles Ashbacher, A. R. Boyd, Scott H. Brown, Paul S. Bruckman, Herta T. Freitag, Ray Melham, Ioan Sadoveanu, Bob Prielipp, and the proposer.

Pell Relations

<u>B-695</u> *Proposed by Russell Euler, Northwest Missouri State U., Maryville, MO*

Define the sequences $\{P_n\}$ and $\{Q_n\}$ by

$$P_0 = 0,\ P_1 = 1,\ P_{n+2} = 2P_{n+1} + P_n \quad \text{for } n \geq 0$$

and

$$Q_0 = 1,\ Q_1 = 1,\ Q_{n+2} = 2Q_{n+1} + Q_n \quad \text{for } n \geq 0.$$

Find a simple formula expressing Q_n in terms of P_n.

Solution 1 by Hans Kappus, Rodersdorf, Switzerland

Let

$$p = 1 + \sqrt{2} \quad \text{and} \quad q = 1 - \sqrt{2}$$

be the roots of the characteristic equation $t^2 - 2t - 1 = 0$ so that the Binet form for the elements of the sequences are given by

$$P_n = \frac{p^n - q^n}{2\sqrt{2}} \quad \text{and} \quad Q_n = \frac{p^n + q^n}{2}.$$

Squaring the second relation, subtracting twice the square of the first relation, and observing that $pq = -1$ yields

$$Q_n^2 - 2P_n^2 = (-1)^n.$$

From the initial conditions and the recurrence, we see that $Q_n > 0$ for all n. Hence, the desired formula is

$$Q_n = \sqrt{2P_n^2 + (-1)^n}.$$

This was the formula expected by the editor. Hendel, however, found perhaps a simpler formula which we now present.

Solution 2 by Russell Jay Hendel, Dowling College, Oakdale, NY

For $n > 0$, we have

$$Q_n = \{P_n\sqrt{2}\}$$

where $\{x\}$ denotes the integer nearest to x.

Proof: Using the Binet forms as found in Solution 1, we find

$$|Q_n - P_n\sqrt{2}| = |q^n| = |(1-\sqrt{2})^n| < \frac{1}{2}$$

for $n \geq 1$, from which the result follows. The formula may also be written as

$$Q_n = \left\lfloor P_n\sqrt{2} + \frac{1}{2}\right\rfloor, \quad \text{for } n > 0,$$

where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. This formula follows also as a particular case of Problem B-680.

Beasley found the formula $Q_n = (1+\sqrt{2})^n - P_n\sqrt{2}$.

In Lucas' seminal paper of 1878 [1], he investigated two similar recurrences $\{U_n\}$ *and* $\{V_n\}$ *defined by*

$$U_{n+2} = PU_{n+1} - QU_n, \; V_{n+2} = PV_{n+1} - QV_n,$$

$$U_0 = 0, \; U_1 = 1, \; V_0 = 2, \; V_1 = P.$$

Lucas showed (page 199) that the two sequences are related by

$$V_n^2 - \Delta U_n^2 = 4Q^n$$

where $\Delta = P^2 - 4Q$.

Many readers interpreted the problem differently. Instead of expressing Q_n *in terms of* P_n *for a given* n, *they showed how to relate the sequence* $\{Q_n\}$ *in terms of the sequence* $\{P_n\}$. *The following solution uses this interpretation.*

Solution 3 by Glenn Bookhout, N. Carolina Wesleyan Col., Rocky Mount, NC

Define the sequence $\{T_n\}$ by $T_0 = 1$, $T_1 = 0$, and

$$T_{n+2} = 2T_{n+1} + T_n$$

for $n \geq 0$. Then, clearly, $Q_n = P_n + T_n$ for all n. But $T_2 = 1$. Thus $T_n = P_{n-1}$ for $n > 0$. Hence the desired formula is $Q_n = P_n + P_{n-1}$.

Another such formula found by several of our solvers was $Q_n = P_{n+1} - P_n$. *Popov gave* $P_n = (Q_n + Q_{n-1})/2$. *The numbers* P_n *and* Q_n *are known as Pell numbers (of the first and second kind) and the relation* $Q_n = P_n + P_{n-1}$ *is well known. Sadoveanu generalized the problem to two sequences* $\{P_n\}$ *and* $\{Q_n\}$ *defined by*

$$P_0 = 0, \; P_1 = 1, \; P_{n+2} = aP_{n+1} + bP_n$$

and

$$Q_0 = 1, \; Q_1 = 1, \; Q_{n+2} = aQ_{n+1} + bQ_n$$

where a *and* b *are arbitrary constants. In this case, a simple induction argument shows that the two sequences are related by the formula*

$$Q_n = P_n + bP_{n-1}, \quad \text{for } n > 0.$$

Reference

1. Edouard Lucas. "Théorie des fonctions numériques simplement périodiques." *American Journal of Mathematics* 1 (1878):184-240, 289-321.

Also solved by Charles Ashbacher, Paul S. Bruckman, Chris Clark & H. K. Krishnapriyan, Herta T. Freitag, Pentti Haukkanen, Joe Howard, Carl Libis, Ray Melham, Blagoj S. Popov. Bob Prielipp, Ioan Sadoveanu, H.-J. Seiffert, Lawrence Somer, and the proposer (two solutions).

A Nonprimitive Pythagorean Triple

B-696 *Proposed by Herta T. Freitag, Roanoke, VA*

Let (a, b, c) be a Pythagorean triple with the hypotenuse $c = 5F_{2n+3}$ and $a = L_{2n+3} + 4(-1)^{n+1}$.

(a) Determine b.
(b) For what values of n, if any, is the triple primitive? [The elements of a primitive triple have no common factor.]

Solution to part (a) by Paul S. Bruckman, Edmonds, WA

From the well-known formulas (Identities 5 and 23 in [1]),

$$5F_a = L_{a-1} + L_{a+1} \quad \text{and} \quad L_{2a} - 2(-1)^a = 5F_a,$$

we see that $c = L_{2n+4} + L_{2n+2}$. Hence,

$$c + a = L_{2n+4} + L_{2n+3} + L_{2n+2} - 4(-1)^{n+2}$$
$$= 2(L_{2n+4} - 2(-1)^{n+2}) = 10F_{n+2}^2.$$

Also,

$$c - a = L_{2n+4} - L_{2n+3} + L_{2n+2} - 4(-1)^{n+1}$$
$$= 2(L_{2n+2} - 2(-1)^{n+1}) = 10F_{n+1}^2.$$

Then,

$$b^2 = c^2 - a^2 = (c + a)(c - a) = 10^2F_{n+2}^2F_{n+1}^2,$$

so $b = 10F_{n+1}F_{n+2}$.

Reference

1. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section*. Ellis Harwood Ltd., West Sussex, England, 1989.

Several solvers found the equivalent formula

$$b = 2(L_{2n+3} + (-1)^n) = 2a + 10(-1)^n.$$

Beasley found $b = 5(F_{2n+3} - F_n^2)$.

Solution to part (b) by Brian D. Beasley, Presbyterian College, Clinton, SC

From $c = 5F_{2n+3}$ and $b = 10F_{n+1}F_{n+2}$, it is clear that $5|c$ and $5|b$. Thus, $5^2|(c^2 - b^2)$ or $5^2|a^2$, which implies that $5|a$. Therefore, the triple is never primitive.

To show that a Pythagorean triple is primitive, it is sufficient to show that no two elements have a common factor. Most solvers showed that $5|a$ using congruences or by finding explicit representations for a. Bruckman showed that

$$a = 5(L_{2n+3} - 2F_{n+1}F_{n+2}).$$

Somer showed that

$$a = 5(F_{n+2}^2 - F_{n+1}^2).$$

Also solved by Charles Ashbacher, Brian D. Beasley, Paul S. Bruckman, Russell Jay Hendel, Nicola Lisi, Bob Prielipp, H.-J. Seiffert, Lawrence Somer, and the proposer.

A Sum of Quotients

B-697 *Proposed by Richard André-Jeannin, Sfax, Tunisia*

Find a closed form for the sum

$$S_n = \sum_{k=1}^{n} \frac{q^{k-1}}{w_k w_{k+1}}$$

where $w_n \neq 0$ for all n and $w_n = pw_{n-1} - qw_{n-2}$ for $n \geq 2$, with p and q nonzero constants.

Solution by A. P. Hillman, Albuquerque, NM

Let $\{u_n\}$ be the sequence defined by

$$u_n = pu_{n-1} - qu_{n-2}, \text{ with } u_0 = 0 \text{ and } u_1 = 1.$$

Let

$$D_n = \begin{vmatrix} u_n & u_{n+1} \\ w_{n+1} & w_{n+2} \end{vmatrix} = \begin{vmatrix} u_n & pu_n - qu_{n-1} \\ w_{n+1} & pw_{n+1} - qw_n \end{vmatrix}.$$

Subtracting p times the first column of this last determinant from the second column shows that $D_n = qD_{n-1}$. Repeated application of this formula yields

$$D_n = q^n D_0.$$

Since

$$D_0 = \begin{vmatrix} 0 & 1 \\ w_1 & w_2 \end{vmatrix} = -w_1,$$

we find that

$$u_n w_{n+2} - u_{n+1} w_{n+1} = -q^n w_1 \quad \text{or} \quad \frac{u_n}{w_{n+1}} - \frac{u_{n+1}}{w_{n+2}} = -w_1 \frac{q^n}{w_{n+1} w_{n+2}}.$$

Thus,

$$S_n = \sum_{k=1}^{n} \frac{q^{k-1}}{w_k w_{k+1}}$$

$$= -\frac{1}{w_1}\left[\left(\frac{u_0}{w_1} - \frac{u_1}{w_2}\right) + \left(\frac{u_1}{w_2} - \frac{u_2}{w_3}\right) + \cdots + \left(\frac{u_{n-1}}{w_n} - \frac{u_n}{w_{n+1}}\right)\right] = \frac{u_n}{w_1 w_{n+1}}.$$

Strictly speaking, the answer u_n/w_1w_{n+1} is not in closed form since it involves the term u_n which is defined via a recurrence. However, we can give a closed form expression for u_n by the well-known Binet formula:

$$u_n = \frac{x^n - y^n}{x - y}$$

where x and y are the roots of the characteristic equation $t^2 - pt + q = 0$.

Other equivalent formulas were found by some solvers. For example, Sadoveanu found

$$S_n = \frac{1}{q(w_1 - xw_0)}\left[\frac{x}{w_1} - \frac{x^{n+1}}{w_{n+1}}\right] \quad \text{and} \quad S_n = \frac{1}{q(w_1 - yw_0)}\left[\frac{y}{w_1} - \frac{y^{n+1}}{w_{n+1}}\right].$$

Kappus found

$$S_n = \frac{1}{d}\left[\frac{w_n}{w_{n+1}} - \frac{w_0}{w_1}\right]$$

where $d = w_1^2 - w_0w_2$, *providing that* $d \neq 0$. *Also assuming* $d \neq 0$, *Popov found*

$$S_n = \frac{1}{qd}\left[\frac{w_2}{w_1} - \frac{w_{n+2}}{w_{n+1}}\right],$$

which generalizes the formula that Lucas found in 1878 for the special case in which $d = 1$ *(see page 196 of [1] or page 18 of [2]).*

References

1. Edouard Lucas. "Théorie des fonctions numériques simplement périodiques." *American Journal of Mathematics* *1* (1878):184-240, 289-321.
2. Edouard Lucas. *The Theory of Simply Periodic Numerical Functions*. The Fibonacci Association, 1969.

Also solved by Paul S. Bruckman, Russell Jay Hendel, Hans Kappus, Blagoj S. Popov, Ioan Sadoveanu, and the proposer.

Late solution to B-684 by Nicola Lisi.

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