WARING'S FORMULA, THE BINOMIAL FORMULA, AND GENERALIZED FIBONACCI MATRICES

Piero Filipponi
Fondazione Ugo Bordoni, Rome, Italy 00142
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1. Introduction

Fibonacci matrices are square matrices the entries of the successive powers of which are related to Fibonacci numbers: the most celebrated among them is the 2-by-2 Q-matrix [1].

In previous papers (e.g., see [3] and [6]) properties of the *generalized Fibonacci Q-matrix*, denoted by M and defined as

(1.1) $\quad M = \begin{bmatrix} m & 1 \\ 1 & 0 \end{bmatrix}$ (m a positive integer),

have been used to evaluate infinite sums involving the *generalized Fibonacci* (U_n) and *Lucas* (V_n) *numbers*

(1.2) $\quad U_n = mU_{n-1} + U_{n-2}, \quad (U_0 = 0,\ U_1 = 1),$

(1.3) $\quad V_n = mV_{n-1} + V_{n-2}, \quad (V_0 = 2,\ V_1 = m).$

Note that when $m = 1$, M is the Q-matrix of [1] so that U_n and V_n are the traditional Fibonacci and Lucas numbers.

The aim of this paper is to show how, using M, M^{-1}, and some other matrices related to M, we can evaluate a variety of finite sums involving U_n and/or V_n. The underlying idea consists in using 2-by-2 *commuting* Fibonacci matrices (say, A and B) so that the matrix analogues of the *binomial formula*

(1.4) $$(A + B)^n = \sum_{j=0}^{n} \binom{n}{j} A^j B^{n-j}$$

and of the *Waring formula* (e.g., see [2], formula (1.2))

(1.5) $$A^n + B^n = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{n}{n-j} \binom{n-j}{j} (AB)^j (A+B)^{n-2j} \quad (n > 0),$$

where the symbol $\lfloor \cdot \rfloor$ denotes the greatest integer function, remain valid. The Fibonacci-type identities are then established by equating the corresponding entries of the matrices on the right-hand side (rhs) and left-hand side (lhs) of (1.4) and (1.5). Most of the identities worked out in this paper as examples of the use of this technique are believed to be new.

Throughout this paper, boldface letters always denote matrices; for example, I denotes the 2-by-2 identity matrix.

2. Definitions

First, we recall that the numbers U_n and V_n can be expressed in closed form by means of the *Binet forms*

(2.1) $\quad U_n = (\alpha^n - \beta^n)/\Delta$

and

(2.2) $\quad V_n = \alpha^n + \beta^n,$

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where

$$(2.3)\qquad \begin{cases} \Delta = \sqrt{m^2 + 4} \\ \alpha = (m + \Delta)/2 \\ \beta = (m - \Delta)/2. \end{cases}$$

Observe that

$$(2.4)\qquad U_{n-1} + U_{n+1} = V_n;$$

identity (2.4) will be widely used throughout the algebraic manipulations without specific reference.

Then, we recall that (e.g., see [6])

$$(2.5)\qquad M^n = \begin{bmatrix} U_{n+1} & U_n \\ U_n & U_{n-1} \end{bmatrix} \quad (n \ge 0)$$

and

$$(2.6)\qquad M^{-1} = M - mI = \begin{bmatrix} 0 & 1 \\ 1 & -m \end{bmatrix}.$$

From formula (2.32) of [6], it is readily seen that

$$(2.7)\qquad (M^{-1})^n = M^{-n} = (-1)^n \begin{bmatrix} U_{n-1} & -U_n \\ -U_n & U_{n+1} \end{bmatrix} \quad (n \ge 0).$$

Finally, let us define the following 2-by-2 matrices.

$$(2.8)\qquad H = R(1) = M + M^{-1} = \begin{bmatrix} m & 2 \\ 2 & -m \end{bmatrix} = 2M - mI,$$

$$(2.9)\qquad M^n + M^{-n} = R(n) = \begin{cases} V_n I & (n \text{ even}), \\ U_n H & (n \text{ odd}). \end{cases}$$

Using formulas (2.24)-(2.27) in [6] and taking into account that the eigenvalues of H are $\lambda_1 = \Delta$ and $\lambda_2 = -\Delta$, after some simple manipulations we obtain

$$(2.10)\qquad H^n = \begin{cases} \Delta^n I & (n \text{ even}) \\ \Delta^{n-1} H & (n \text{ odd}) \end{cases}$$

and

$$(2.11)\qquad H^{-n} = H^n/\Delta^{2n}.$$

Since the matrices (2.5)-(2.11) are polynomials in M, they commute so that they can replace A and B in (1.4) and (1.5) above and can be used to obtain Fibonacci-type identities, as will be shown in sections 3 and 4.

3. Use of the Binomial Formula

In this section we give some examples of the use of the matrices defined in section 2 in connection with (1.4).

Example 1: Using (2.6), we can write

$$(3.1)\qquad M^n = (M^{-1} + mI)^n = \sum_{j=0}^{n} \binom{n}{j} M^{-j} (mI)^{n-j} = m^n \sum_{j=0}^{n} \binom{n}{j} (mM)^{-j}.$$

Equating the upper right-hand entries of the matrices on the lhs and rhs of (3.1), by (2.5) and (2.7), we can write

$$\sum_{j=0}^{n} \binom{n}{j} \left(-\frac{1}{m}\right)^j U_j = -U_n/m^n$$

whence, recalling that $U_0 = 0$,

$$(3.2) \qquad \sum_{j=1}^{n-1} \binom{n}{j}\left(-\frac{1}{m}\right)^j U_j = \begin{cases} -2U_n/m^n & (n \text{ even}), \\ 0 & (n \text{ odd}). \end{cases}$$

Example 2: Using (2.8) and (2.6) and omitting the intermediate steps, we can write

$$H^n = (2M - mI)^n = (-m)^n \sum_{j=0}^{n} \binom{n}{j}\left(-\frac{2}{m}\right)^j M^j.$$

Using reasoning similar to the preceding [cf. (2.5) and (2.10)] we obtain

$$(3.3) \qquad \sum_{j=1}^{n} \binom{n}{j}\left(-\frac{2}{m}\right)^j U_j = \begin{cases} 0 & (n \text{ even}), \\ -2\Delta^{n-1}/m^n & (n \text{ odd}). \end{cases}$$

Example 3: From (2.9), we have

$$R^k(n) = \sum_{j=0}^{k} \binom{k}{j} M^{n(2j-k)}$$

whence, after some manipulations, we can write

$$R^k(n) = \begin{cases} \binom{k}{k/2} I + \sum_{j=0}^{(k-2)/2} \binom{k}{j} R(n(k-2j)) & (k \text{ even}), \\ \sum_{j=0}^{(k-1)/2} \binom{k}{j} R(n(k-2j)) & (k \text{ odd}). \end{cases}$$

Equating the upper-left entries of the matrices on the lhs and rhs of (3.4) and taking (2.9) and (2.10) into account, we obtain

$$(3.5) \qquad \sum_{j=0}^{(k-2)/2} \binom{k}{j} V_{n(k-2j)} = V_n^k - \binom{k}{k/2} \quad (k \text{ and } n \text{ even}),$$

$$(3.6) \qquad \sum_{j=0}^{(k-2)/2} \binom{k}{j} V_{n(k-2j)} = U_n^k \Delta^k - \binom{k}{k/2} \quad (k \text{ even}, n \text{ odd}),$$

$$(3.7) \qquad \sum_{j=0}^{(k-1)/2} \binom{k}{j} V_{n(k-2j)} = V_n^k \quad (k \text{ odd}, n \text{ even}),$$

$$(3.8) \qquad \sum_{j=0}^{(k-1)/2} \binom{k}{j} U_{n(k-2j)} = U_n^k \Delta^{k-1} \quad (k \text{ and } n \text{ odd}).$$

Example 4: From (2.6), let us write

$$(3.9) \qquad m^n I = (M - M^{-1})^n = (-1)^n \sum_{j=0}^{n} \binom{n}{j} (-1)^j M^{2j-n}$$

$$= \begin{cases} \binom{n}{n/2} (-1)^{n/2} I + \sum_{j=0}^{(n-2)/2} \binom{n}{j} (-1)^j R(n-2j) & (n \text{ even}), \\ \sum_{j=0}^{(n-1)/2} \binom{n}{j} (-1)^j [M^{n-2j} - M^{-(n-2j)}] & (n \text{ odd}). \end{cases}$$

Equating the upper-left entries of the matrices on the lhs and rhs of (3.9) and taking (2.9), (2.5), and (2.7) into account, yields

$$(3.10)\quad \sum_{j=0}^{[(n-1)/2]} \binom{n}{j}(-1)^j V_{n-2j} = \begin{cases} m^n - \binom{n}{n/2}(-1)^{n/2} & (n \text{ even}), \\ m^n & (n \text{ odd}). \end{cases}$$

Example 5: Now let us consider the matrix

$$(3.11)\quad HM^n = \begin{bmatrix} V_{n+1} & V_n \\ V_n & V_{n-1} \end{bmatrix} \quad [\text{from } (2.5) \text{ and } (2.8)]$$

and recall [see (2.10)] that

$$(3.12)\quad (HM)^n = H^n M^n = \begin{cases} \Delta^n M^n & (n \text{ even}), \\ \Delta^{n-1} HM^n & (n \text{ odd}). \end{cases}$$

From (2.8) we have $HM = M^2 + I$ so that we can write

$$(3.13)\quad (HM)^n = (M^2 + I)^n = \sum_{j=0}^{n} \binom{n}{j} M^{2j}.$$

Equating the upper-right entries of the matrices on the lhs and rhs of (3.13) and taking (3.12), (2.5), and (3.11) into account gives

$$(3.14)\quad \sum_{j=0}^{n} \binom{n}{j} U_{2j} = \begin{cases} \Delta^n U_n & (n \text{ even}), \\ \Delta^{n-1} V_n & (n \text{ odd}). \end{cases}$$

Moreover, from (2.11) and (2.7), after some manipulations it can be seen that

$$(3.15)\quad (HM^n)^{-1} = H^{-1}M^{-n} = \frac{(-1)^n}{\Delta^2}\begin{bmatrix} -V_{n-1} & V_n \\ V_n & -V_{n+1} \end{bmatrix}.$$

From (3.11) and (3.15), equating the upper-left entries of the matrices on the lhs and rhs of the matrix equation $(HM^n)^{-1}HM^n = I$, yields

$$(3.16)\quad V_n^2 - V_{n-1}V_{n+1} = \Delta^2(-1)^n.$$

Finally, from (3.11) and (2.7) let us write

$$(3.17)\quad H = (-1)^n \begin{bmatrix} V_{n+1} & V_n \\ V_n & V_{n-1} \end{bmatrix}\begin{bmatrix} U_{n-1} & -U_n \\ -U_n & U_{n+1} \end{bmatrix}.$$

If we equate the entries of H and those of the matrix product on the rhs of (3.17), by (2.8) we can write

$$(3.18)\quad U_{n-1}V_{n+1} - U_nV_n = m(-1)^n \quad (\text{upper-left entry}),$$

$$(3.19)\quad U_{n+1}V_n - U_nV_{n+1} = 2(-1)^n \quad (\text{upper-right entry}).$$

From (3.18) and (3.19), the following identities involving *Pell numbers* (P_n, i.e., U_n with $m = 2$) and *Pell-Lucas numbers* (Q_n, i.e., V_n with $m = 2$) [5] can be immediately established:

$$(3.20)\quad \begin{cases} Q_{n+1}(P_{n-1} + P_n) = Q_n(P_n + P_{n+1}) \\ P_n(Q_{n+1} - Q_n) = P_{n+1}Q_n - P_{n-1}Q_{n+1}. \end{cases}$$

4. Use of the Waring Formula

In this section we give a few examples of use of the matrices defined in section 2 in connection with (1.5). Some simple congruencial properties of the numbers U_n and V_n are then established on the basis of the identity obtained in the first of the given examples.

Example 6: By (1.5) and (2.8) we can write

$$(4.1) \qquad R(n) = M^n + M^{-n} = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{n}{n-j} \binom{n-j}{j} H^{n-2j} \quad (n > 0).$$

By equating the upper-left entries on the lhs and rhs of (4.1) and taking (2.9) and (2.10) into account, we obtain a rather curious formula valid [see (1.5)] for $n > 0$, namely

$$(4.2) \qquad \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{n}{n-j} \binom{n-j}{j} (\Delta^2)^{\lfloor n/2 \rfloor - j} = \begin{cases} V_n & (n \text{ even}), \\ U_n & (n \text{ odd}). \end{cases}$$

The curiousness of (4.2) lies in the fact that analogous formulas (e.g., see [4] formulas (1.6) and (1.7)) give *separately* all numbers U_n and V_n (independently of the parity of n).

Example 7: By (4.1) let us write

$$(4.3) \qquad M^{2n} + I = M^n(M^n + M^{-n}) = M^n R(n)$$
$$= \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{n}{n-j} \binom{n-j}{j} M^n H^{n-2j} \quad (n > 0)$$

and observe [see (2.5), (2.8), and (2.10)] that the upper-left entry x_{11} of $M^n H^{n-2j}$ is

$$(4.4) \qquad x_{11} = \begin{cases} U_{n+1}\Delta^{n-2j} & (n \text{ even}), \\ V_{n+1}\Delta^{n-1-2j} & (n \text{ odd}). \end{cases}$$

Equating the upper-left entries of the matrices on the lhs and rhs of (4.3) and taking (2.5) and (4.4) into account gives an identity the lhs of which is the same as that of (4.2), while its rhs equals $U_{2n+1} + 1$ divided by either U_{n+1} (n even) or V_{n+1} (n odd). Comparing these identities with (4.2) yields

$$(4.5) \qquad U_{2n+1} + 1 = \begin{cases} U_{n+1}V_n & (n \text{ even}), \\ U_n V_{n+1} & (n \text{ odd}). \end{cases}$$

Observe that (4.5) also holds for $n = 0$.

Example 8: Let us replace A by mM and B by I in (1.5) and take into account that, from (1.1), (1.2), and (2.5), we have

$$(4.6) \qquad mM + I = M^2.$$

Equating the upper-right entries of the matrices on the lhs and rhs of the so-obtained matrix identity yields

$$(4.7) \qquad \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{n}{n-j} \binom{n-j}{j} m^j U_{2n-3j} = m^n U_n \quad (n > 0).$$

Observe that using the matrix identity $(mM)^n = (M^2 - I)^n$ [directly derived from (4.6)] in connection with (1.4) gives an alternative expression for the rhs of (4.7), namely,

$$(4.8) \qquad (-1)^n \sum_{j=0}^{n} (-1)^j \binom{n}{j} U_{2j} = m^n U_n.$$

4.1 Some Congruencial Properties of U_n and V_n

Some congruencial properties of U_n and V_n can be derived easily from (4.2). If $p > 2$ is a prime, then from (4.2) we can write

$$U_p = \Delta^{p-1} + \sum_{j=1}^{(p-1)/2} (-1)^j \frac{p}{p-j}\binom{p-j}{j}\Delta^{p-1-2j} \tag{4.9}$$

whence, noting that the sum on the rhs of (4.9) is divisible by p and recalling that $\Delta^2 = m^2 + 4$, we obtain the congruence

$$U_p \equiv (m^2 + 4)^{(p-1)/2} \pmod{p}. \tag{4.10}$$

Equivalently, we can state that $U_p \equiv 0 \pmod{p}$ if $m^2 + 4 \equiv 0 \pmod{p}$ while $U_p \equiv 1$ (or -1) (mod p) if $m^2 + 4$ is (is not) a quadratic residue modulo p.

For n an arbitrary positive odd integer, let us rewrite (4.2) as

$$\begin{aligned} U_n &= (-1)^{(n-1)/2} \frac{2n}{n+1}\binom{(n+1)/2}{(n-1)/2} + \sum_{j=0}^{(n-3)/2} (-1)^j \frac{n}{n-j}\binom{n-j}{j}\Delta^{n-1-2j} \\ &= n(-1)^{(n-1)/2} + \sum_{j=0}^{(n-3)/2} (-1)^j \frac{n}{n-j}\binom{n-j}{j}\Delta^{n-1-2j} \quad (n \text{ odd}). \end{aligned} \tag{4.11}$$

From (4.11), the congruence

$$U_n \equiv n(-1)^{(n-1)/2} \pmod{m^2 + 4} \quad (n \text{ odd}) \tag{4.12}$$

is immediately obtained. Using the same procedure, for n even we get

$$V_n \equiv 2(-1)^{n/2} \pmod{m^2 + 4} \quad (n \text{ even}). \tag{4.13}$$

5. Conclusions and Further Examples

In this paper it has been shown that a large number of Fibonacci-type identities can be established by using matrices related to M in connection with the binomial formula and the Waring formula. We do believe that matrices other than those defined in section 2 can be employed to obtain further identities.

On the other hand, we wish to point out that the technique discussed in section 2 can also be used profitably in connection with other formulas. For example, consider the matrix equation

$$A^n + B^n = (A + B)\sum_{j=1}^{n} (-1)^{j-1} A^{n-j} B^{j-1}, \tag{5.1}$$

which is valid if $AB = BA$ and n is odd, and the matrix equation

$$\sum_{j=0}^{n} A^j = (A^{n+1} - I)(A - I)^{-1}, \tag{5.2}$$

which is valid if all eigenvalues of A are different from 1.

If we replace A by M and B by M^{-1} in (5.1), we can write

$$\begin{aligned} R(n) &= M^n + M^{-n} = (M + M^{-1})\sum_{j=1}^{n} (-1)^{j-1} M^{n-2j+1} \\ &= H\sum_{j=1}^{n} (-1)^{j-1} M^{n-2j+1} = H\left[(-1)^{(n-1)/2} I + \sum_{j=1}^{(n-1)/2} (-1)^{j-1} R(n - 2j + 1)\right] \end{aligned}$$

and, by (2.9),

$$R(n) = (-1)^{(n-1)/2} H + H\sum_{j=1}^{(n-1)/2} (-1)^{j-1} V_{n-2j+1} I \quad (n \text{ odd}). \tag{5.3}$$

Equating the upper-left entries of the matrices on the lhs and rhs of (5.3) and taking (2.9) and (2.8) into account yields

$$mU_n = (-1)^{(n-1)/2}m + m\sum_{j=1}^{(n-1)/2}(-1)^{j-1}V_{n-2j+1},$$

whence

$$(5.4)\qquad \sum_{j=1}^{(n-1)/2}(-1)^{j-1}V_{n-2j+1} = U_n - (-1)^{(n-1)/2} \quad (n \text{ odd}).$$

Of course, the lhs of (5.4) is an alternating sum of alternate V_k. Since

$$V_k = U_{k-1} + U_{k+1},$$

the sum obviously telescopes so that (5.4) has a more direct derivation.

If we replace A by HM in (5.2) and take (2.8) and (3.12) into account, for n odd we can write

$$(5.5)\qquad \sum_{j=0}^{n}(HM)^j = (H^{n+1}M^{n+1} - I)(HM - I)^{-1} = (H^{n+1}M^{n+1} - I)M^{-2}$$
$$= (\Delta^{n+1}M^{n+1} - I)M^{-2}.$$

Again, by (3.12) and (2.8), the lhs of (5.5) can be rewritten as

$$(5.6)\qquad \sum_{j=0}^{n}(HM)^j = \sum_{j=0}^{(n-1)/2}\Delta^{2j}(M^{2j} + HM^{2j+1}) = \sum_{j=0}^{(n-1)/2}\Delta^{2j}(M^{2j+2} + 2M^{2j}).$$

Equating the upper-left entries of the matrices on the rhs of (5.6) and (5.5) and using (2.7) yields

$$(5.7)\qquad \sum_{j=0}^{(n-1)/2}\Delta^{2j}(U_{2j+3} + 2U_{2j+1}) = \Delta^{n+1}U_n - 1 \quad (n \text{ odd}).$$

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