GENERATION OF GENOCCHI POLYNOMIALS OF FIRST ORDER BY RECURRENCE RELATIONS

A. F. Horadam
The University of New England, Armidale, N.S.W., Australia
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1. Motivation

Genocchi polynomials of the first order, $G_n(x)$, are defined [3] by

$$(1.1)\qquad \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!} = \frac{2t}{e^t+1}e^{tx}$$

as an extension of Genocchi numbers G_n defined in [1].

Following a suggestion by the referee of [3], I show briefly how $G_{2n+1}(x)$ $(n \geq 1)$ may be generated by $x^2 - x = x(x-1)$. Such a possibility is to be expected since by (2.2) $x = 0$ and $x = 1$ are zeros of $G_{2n+1}(x)$. For example,

$$(1.2)\qquad \begin{aligned} G_{13}(x) &= 13[x^{12} - 6x^{11} + 55x^9 - 396x^7 + 1683x^5 - 3410x^3 + 2073x] \\ &= 13[(x^2-x)^6 - 15(x^2-x)^5 + 135(x^2-x)^4 - 736(x^2-x)^3 \\ &\qquad + 2073(x^2-x)^2 - 2073(x^2-x)]. \end{aligned}$$

It is the main purpose of this article to establish an algorithm for deriving a result like (1.2). Equations (3.6) and (3.7) are in fact the *recurrence relations* sought for $G_{2n+1}(x)$, the Genocchi polynomials of odd order. Similarly, we obtain (3.11), a recurrence relation for $G_{2n}(x)$ of even order. Our treatment, which was excluded from [3] because of the already considerable length of that paper, follows that given in [8] for Euler polynomials $E_n(x)$.

The theory expounded here does not generalize to $G_n^{(k)}(x)$, the Genocchi polynomials of order k [3]. An examination of the $G_n^{(k)}(x)$ listed in [3] will readily reveal why this is so.

Another purpose of this article is to answer a question raised at the 1990 International Fibonacci Conference at Wake Forest University, U.S.A.

2. Some Genocchi Formulas

Properties of $G_n(x)$ required to obtain the recurrence relations include [3]

$$(2.1)\qquad \frac{dG_n(x)}{dx} = nG_{n-1}(x), \quad n \geq 1,$$

and

$$(2.2)\qquad G_{2n}\left(\frac{1}{2}\right) = G_{2n+1}(0) = G_{2n+1}(1) = 0, \quad n \geq 1.$$

It is to be noted that

$$(2.3)\qquad G_n(x) = nE_{n-1}(x),$$

from which we have *Genocchi's theorem* ([1], [3], [4])

$$(2.4)\qquad G_{2n} = 2nE_{2n-1}(0)$$

for *Genocchi numbers* $G_n \equiv G_n(0)$ given in [1], [3], and [4] (see [2] also).

However, $E_{2n-1}(0)$ are not Euler numbers, but numbers related to Euler numbers ([3], [5]). Information on Euler polynomials and Bernoulli polynomials may be found, for example, in [5]. Other material of interest relating these polynomials to angular momentum traces occurs in [6], [7], and [8].

3. The Genocchi Generation

Using induction [8] as employed in [6] for Bernoulli polynomials, we can show that

$$(3.1)\qquad G_{2n+1}(x) = Y_n(u),$$

where

$$(3.2)\qquad u = x^2 - x \quad \left(\frac{du}{dx} = 2x - 1\right).$$

With the help of (2.1), (2.2), (3.1), and (3.2), from which

$$\left(\frac{du}{dx}\right)^2 = 4u + 1,$$

we can derive, after a few steps, the differential equation

$$(3.3)\qquad (4u + 1)\frac{d^2Y_n(u)}{du^2} + 2\,\frac{dY_n(u)}{du} = 2n(2n + 1)Y_{n-1}(u).$$

Now let

$$(3.4)\qquad G_{2n+1}(x) = Y_n(u) = \sum_{i=0}^{n} A_i u^i = (2n + 1)\sum_{i=0}^{n} C_i u^i$$

and

$$(3.5)\qquad G_{2n-1}(x) = Y_{n-1}(u) = \sum_{i=0}^{n-1} B_i u^i = (2n - 1)\sum_{i=0}^{n-1} D_i u^i$$

so that, by (2.3), the C_i and D_i are the same as for $E_{2n}(x)$ in [8].

Calculation in (3.3) - (3.5) yields (cf. [8])

$$(3.6)\qquad (2n - 1)A_n = (2n + 1)B_{n-1}$$

and

$$(3.7)\qquad i(i + 1)A_{i+1} + 2i(2i - 1)A_i = 2n(2n + 1)B_{i-1},$$

for $1 \le i \le n - 1$, $n \ge 2$.

Solving (3.6) and (3.7) for $n = 1, 2, 3, \ldots$ gives the constants A_i and B_i in the expansions (3.4) and (3.5). Table 1 supplies an appreviated list of these.

From (2.2) and (3.1), it follows that, for $n \ge 1$,

$$(3.8)\qquad Y_n(u) = G_{2n+1}(x) = 0 \text{ when } x = 0, 1, \text{ i.e., } u = 0.$$

Thus, $Y_n(u)$, $n \ge 1$, has no constant term, i.e., $A_0 = 0$. Likewise, $B_0 = 0$.

Consequently, the recurrence relations (3.6) and (3.7) generate $G_{2n+1}(x) = Y_n(u)$, where $G_1(x) = Y_0(u) = 1 = u^0$.

Table 1

Coefficients A_i of $G_{2n+1}(x) = Y_n(u)$

$n \backslash i$	1	2	3	4	5
1	3				
2	-5	5			
3	21	-21	7		
4	-133	133	-54	9	
5	1705	-1705	605	110	11

Note that in (3.7) when $i = 1$, $n \ge 2$ ($B_0 = 0$), we obtain

$$(3.9)\qquad A_2 = -A_1.$$

In Table 2 of [8], we observe the apparently unnoticed fact that the elements in column 2 for the Euler polynomials $E_{2n}(x)$ are the Genocchi numbers G_4, G_6, G_8, G_{10}, ..., while those in column 1 are the negatives of these Genocchi numbers.

Why is this so?

For each $n \geq 2$,

$$
\begin{aligned}
(3.10) \quad G_{2n} &= 2nE_{2n-1}(0) && \text{from (2.4),} \\
&= \frac{d}{dx}E_{2n}(x)\Big|_{x=0} && \text{by (2.1), (2.3),} \\
&= (2x-1)\frac{d}{du}\left\{\sum_{i=0}^{n} C_i u^i\right\}\Big|_{\substack{x=0\\u=0}} && \text{from [8], equation (32)} \\
&= -C_1.
\end{aligned}
$$

Because of (3.9) and (3.10), the elements in the first and second columns of our Table 1 will be appropriate multiples of Genocchi numbers, namely,

$$(2n+1)G_{2n} = -A_1 \quad \text{for each } n \geq 2.$$

Coming now to generators of $G_{2n}(x)$ we have, from (2.1),

$$
\begin{aligned}
(3.11) \quad G_{2n}(x) &= \frac{1}{2n+1}\,\frac{dG_{2n+1}(x)}{dx} \\
&= \frac{2x-1}{2n+1}\,\frac{dY_n(u)}{du} \qquad \text{by (3.1), (3.2),} \\
&= (2x-1)Z_{n-1}(u),
\end{aligned}
$$

i.e.,

$$(3.12) \quad (2n+1)Z_{n-1}(u) = \frac{dY_n(u)}{du}$$

i.e., the $Z_{n-1}(u)$ can be derived from the known $Y_n(u)$.

For example,

$$G_6(x) = 3(2x-1)(u^2-2u+1) = (2x-1)Z_2(u)$$

with

$$\frac{dY_3(u)}{du} = 7\frac{d}{du}(3u-3u^2+u^3) = 7[3(1-2u+u^2)] = 7Z_2(u)$$

on using our Table 1. From this table for $Z_{n-1}(u)$, a corresponding table for $A_{n-1}(u)$ could be constructed.

4. A Question Answered

Consider $x^2 - x - 1 = u - 1$ by (3.2). This is the well-known algebraic expression for the Fibonacci recurrence, $F_{n+2} - F_{n+1} - F_n = 0$, whose zeros are $(1+\sqrt{5})/2$ and its negative reciprocal.

Next, from [1] or (1.1),

$$
(4.1) \quad \begin{cases} G_5(x) = 5u(u-1) \\ G_6(x) = 3(2x-1)(u-1)^2 = 3(u-1)^2\dfrac{du}{dx}, \end{cases}
$$

i.e., the term $u-1$ in $G_5(x)$ is squared in $G_6(x)$.

At my address on Genocchi polynomials to the Fourth International Conference on Fibonacci Numbers and Their Applications held at Wake Forest University in Winston-Salem, North Carolina, U.S.A. (see [3]), I was asked: "Is there any pattern in the $G_n(x)$ for other (positive) powers of $u-1$?"

Assume that, for some N, the Genocchi polynomial $G_N(x)$ contains a factor $(u - 1)^k$. Then, by (2.1), $G_{N-1}(x)$ contains a factor $(u - 1)^{k-1}$.

There are two cases to be investigated, namely,

I. $N = 2n$ and II. $N = 2n + 1$.

Recall that, by virtue of (2.2),

$$\begin{cases} 2x - 1 = \dfrac{du}{dx} \text{ is always a factor of } G_{2n}(x), \\ x(x - 1) = u \text{ is always a factor of } G_{2n+1}(x). \end{cases}$$

<u>Case I</u>. Suppose

$$(\alpha) \qquad G_{2n}(x) = n\frac{du}{dx}(u - 1)^m$$

$$(\beta) \qquad G_{2n-1}(x) = (2n - 1)u(u - 1)^{m-1},$$

the numbers $n = 2n/2$ and $2n - 1$ being necessary coefficients (see [3]). Now

$$\begin{aligned} \frac{dG_{2n}(x)}{dx} &= n\{2(u - 1)^m + (4u + 1)m(u - 1)^{m-1}\} && \text{from } (\alpha) \\ (\gamma) \qquad &= n(u - 1)^{m-1}\{(2 + 4m)u + m - 2\} \\ &= 2nG_{2n-1}(x) && \text{by } (2.1) \\ (\delta) \qquad &= 2n(2n - 1)u(u - 1)^{m-1} && \text{by } (\beta). \end{aligned}$$

For (α) and (β) to be valid, we must have $(\gamma) = (\delta)$. Equating these produces

$$(2 + 4m)u + m - 2 = (4n - 2)u,$$

whence

$$(4.2) \qquad \begin{cases} m = 2 \\ n = 3 \end{cases}.$$

<u>Case II</u>. Secondly, suppose

$$(\alpha') \qquad G_{2n+1}(x) = (2n + 1)u(u - 1)^p$$

$$(\beta') \qquad G_{2n}(x) = \frac{du}{dx}(u - 1)^{p-1}.$$

Then,

$$\begin{aligned} \frac{dG_{2n+1}(x)}{dx} &= (2n + 1)\left\{\frac{du}{dx}(u - 1)^p + up(u - 1)^{p-1}\frac{du}{dx}\right\} && \text{from } (\alpha') \\ (\gamma') \qquad &= (2n + 1)(u - 1)^{p-1}\frac{du}{dx}\{u - 1 + up\} \\ &= (2n + 1)G_{2n}(x) && \text{by } (2.1) \\ (\delta') \qquad &= (2n + 1)n\frac{du}{dx}(u - 1)^{p-1} && \text{by } (\beta'). \end{aligned}$$

Solving (γ') and (δ') leads to $p = n = -1$, which must be discarded because p and n were assumed to be positive.

Cases I and II demonstrate that, by (4.2), the only occurrence of powers of $u - 1$ is that in $G_5(x)$ and $G_6(x)$ given in (4.1).

Our answer to the question is thus: No!

References

1. L. Comtet. *Advanced Combinatorics*. Reidel, 1974.
2. A. Genocchi. "Intorno all'espressione generale de'Numeri Bernulliani. Nota." *Annali di Scienze Matematiche e Fisiche 3* (1852):395-405.
3. A. F. Horadam. "Genocchi Polynomials." *Proceedings of the Fourth International Conference on Fibonacci Numbers and Their Applications*. Kluwer, 1991, pp. 145-66.
4. E. Lucas. *Théorie des Nombres*. Blanchard, 1961.
5. N. E. Nörlund. *Vorlesungen über Differenzenrechnung*. Chelsea, 1954.
6. P. R. Subramanian. "A Short Note on the Bernoulli Polynomial of the First Kind." *Math. Student 42* (1974):47-59.
7. P. R. Subramanian & V. Devanathan. "Recurrence Relations for Angular Momentum Traces." *J. Phys. A.: Math. Gen. 13* (1980):2689-93.
8. P. R. Subramanian & V. Devanathan. "Generation of Angular Momentum Traces and Euler Polynomials by Recurrence Relations." *J. Phys. A.: Math. Gen. 18* (1985):2909-15.

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