

A COMBINATORIAL PROBLEM IN THE FIBONACCI NUMBER SYSTEM  
AND TWO-VARIABLE GENERALIZATIONS OF  
CHEBYSHEV'S POLYNOMIALS

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*To my mother on the occasion of her 70th birthday.*

1. Summary

We consider the following three-term recursion formula

$$(1.1a) \quad S_{-1} = 0, \quad S_0 = 1$$

$$(1.1b) \quad S_n = Y(n)S_{n-1} - S_{n-2}, \quad n \geq 1$$

$$(1.1c) \quad Y(n) = Yh(n) + y(1 - h(n)),$$

where  $h(n)$  is the  $n^{\text{th}}$  digit of the Fibonacci-"word" 1011010110... given explicitly by (see [7], [11], [9], [20], [19])

$$(1.2) \quad h(n) = [(n+1)\phi] - [n\phi] - 1,$$

where  $[a]$  denotes the integer part of a real number  $a$ , and

$$\phi := (1 + \sqrt{5})/2,$$

obeying  $\phi^2 = \phi + 1$ ,  $\phi > 1$ , is the golden section [10], [9], [4].

For  $Y = y$  one recovers Chebyshev's  $S_n(y)$  polynomials of degree  $n$  [1]. In the general case certain two-variable polynomials  $S_n(Y, y)$  emerge.

The theory of continued fractions (see [18]) shows that  $(-i)^n S_n(Y, y)$  can be identified with the denominator of the  $n^{\text{th}}$  approximation of the regular continued fraction ( $i^2 = -1$ )

$$(1.3) \quad [0; -iY(1), -iY(2), \dots, -iY(k), \dots] \\ \equiv 1/(-iY(1) + 1/(-iY(2) + 1/(\dots .$$

The polynomials  $S_n(Y, y)$  can be written as

$$(1.4) \quad S_n(Y, y) = \sum_{\ell=0}^{[n/2]} (-1)^\ell \sum_{k=k_{\min}}^{k_{\max}} (n; \ell, k) Y^{z(n)-\ell-k} y^{n-z(n)-\ell+k},$$

where the coefficients  $(n; \ell, k)$  are defined recursively by

$$(1.5) \quad (n; \ell, k) = (n-1; \ell, k) + (h(n-1) + h(n) - 1)(n-2; \ell-1, k-1) \\ + (2 - h(n-1) - h(n))(n-2; \ell-1, k),$$

with certain input quantities. The range of the  $k$  index is bounded by

$$(1.6a) \quad k_{\min} \equiv k_{\min}(n, \ell) := \max\{0, \ell - (n - z(n))\},$$

$$(1.6b) \quad k_{\max} \equiv k_{\max}(n, \ell) := \min\{z(n) - \ell, \min(\ell, p(n))\},$$

with

$$(1.7) \quad z(n) = \sum_{k=1}^n h(k),$$

$$(1.8) \quad p(n) = \sum_{k=0}^{n-1} (h(k+1) + h(k) - 1).$$

The polynomials  $S_n(Y, y)$  are listed for  $n = 0(1)13$  in Table 1. They are generating functions for the numbers  $(n; \ell, k)$  which are shown to have a combinatorial meaning in the Fibonacci number system. This system is based on the fact that every natural number  $N$  has a unique representation (see [23], [5], [21], [11], [20]) in terms of Fibonacci numbers (see [10] and [4]):

$$(1.9) \quad N = \sum_{i=0}^r s_i F_{i+2}, \quad s_i \in \{0, 1\}, \quad s_i s_{i+1} = 0.$$

(Zeckendorf's representation of the second kind in which one writes the number 1 as  $F_2$  and not as  $F_1$ .)

**Table 1.**  $S_n = Y(n)S_{n-1} - S_{n-2}, S_{-1} = 0, S_0 = 1$   
 $Y(n) = Yh(n) + y(1 - h(n))$   
 $h(n) = [(n + 1)\phi] - [n\phi] - 1$

$n$	$S_n(Y, y)$
0	1
1	$Y$
2	$Yy - 1$
3	$Y(Yy - 2)$
4	$Y^3y - Y(2Y + y) + 1$
5	$Y^3y^2 - Yy(3Y + y) + (2Y + y)$
6	$Y^4y^2 - Y^2y(4Y + y) + 2Y(2Y + y) - 1$
7	$Y^4y^3 - Y^2y^2(5Y + y) + Yy(7Y + 3y) - 2(Y + y)$
8	$Y^5y^3 - Y^3y^2(6Y + y) + Y^2y(11Y + 4y) - 2Y(3Y + 2y) + 1$
9	$Y^6y^3 - Y^4y^2(6Y + 2y) + Y^2y(11Y^2 + 9Yy + y^2) - Y(6Y^2 + 11Yy + 3y^2) + (3Y + 2y)$
10	$Y^6y^4 - Y^4y^3(7Y + 2y) + Y^2y^2(17Y^2 + 10Yy + y^2) - Yy(17Y^2 + 15Yy + 3y^2) + (6Y^2 + 7Yy + 2y^2) - 1$
11	$Y^7y^4 - Y^5y^3(8Y + 2y) + Y^3y^2(23Y^2 + 12Yy + y^2) - Y^2y(28Y^2 + 24Yy + 4y^2) + Y(12Y^2 + 18Yy + 5y^2) - (4Y + 2y)$
12	$Y^8y^4 - Y^6y^3(8Y + 3y) + Y^4y^2(23Y^2 + 19Yy + 3y^2) - Y^2y(28Y^3 + 41Y^2y + 14Yy^2 + y^3) + Y(12Y^3 + 35Y^2y + 20Yy^2 + 3y^3) - (10Y^2 + 9Yy + 2y^2) + 1$
13	$Y^8y^5 - Y^6y^4(9Y + 3y) + Y^4y^3(31Y^2 + 21Yy + 3y^2) - Y^2y^2(51Y^3 + 53Y^2y + 15Yy^2 + y^3) + Yy(40Y^3 + 59Y^2y + 24Yy^2 + 3y^3) - (12Y^3 + 28Y^2y + 14Yy^2 + 2y^3) + (4Y + 3y)$
⋮	

In this number system  $N \hat{=} s_n \dots s_2s_1s_0 \cdot$ , where the dot at the end indicates the  $F_1$  place which is not used.

**Proposition 1:**  $(n; \ell, k)$  gives the number of possibilities to choose, from the natural numbers 1 to  $n$ ,  $\ell$  mutually disjoint pairs of consecutive numbers such that all numbers of  $k$  of these pairs end in the canonical Fibonacci number system in an even number of zeros.

Another formulation is possible if Wythoff's complementary sequences  $\{A(n)\}$  and  $\{B(n)\}$  (see [22], [7], [21], [12], [8], [9], and [4]), defined by

(1.10)  $A(n) := [n\phi]$ ,  $B(n) := [n\phi^2] = n + A(n)$ ,  $n = 1, 2, \dots$ ,  
are introduced.

*Proposition 2:*  $(n; \ell, k)$  is the number of different possibilities to choose, from the numbers  $1, 2, \dots, n$ ,  $\ell$  mutually disjoint pairs of consecutive numbers, say

$$(n_1, n_1 + 1), \dots, (n_\ell, n_\ell + 1) \text{ with } n_j > n_{j-1} + 1 \text{ for } j = 2, \dots, \ell,$$

such that all members of  $k$  pairs among them, say

$$(i_1, i_1 + 1), \dots, (i_k, i_k + 1),$$

are  $A$ -numbers, i.e.,  $i_j = A(m_j)$  and  $i_j + 1 = A(m_j + 1)$  for some  $m_j$  and all  $j = 1, \dots, k$ . For  $\ell = 0$ , put  $(n; 0, 0) = 1$ .

From the analysis of Wythoff's sequences one learns that  $A$ -pairs  $(A(m_j), A(m_j + 1) = A(m_j) + 1)$  occur precisely for  $m_j = B(q_j)$  for some  $q_j \in \mathbb{N}$ . All remaining pairs are either of the  $(A, B)$  or  $(B, A)$  type. Thus, one may state equivalently,

*Proposition 3:*  $(n; \ell, k)$  counts the number of different ways to choose, from the numbers  $1, 2, \dots, n - 1$ ,  $\ell$  distinct nonneighboring numbers such that exactly  $k$  numbers among them, say  $i_1, \dots, i_k$ , are  $AB$ -numbers, i.e., they satisfy for all  $j = 1, \dots, k$ ,  $i_j = A(B(m_j))$  with some  $m_j \in \mathbb{N}$ .

Still another meaning can be attributed to the coefficients of the  $S_n$  polynomials based on the above findings.

*Corollary:* Consider the Zeckendorf representations (with 1 as  $F_2$ ) of the numbers  $0, 1, 2, \dots, F_{n+1} - 1$ . Then exactly  $(n; \ell, k)$  of them need  $\ell$  Fibonacci numbers,  $k$  of which are of the type  $F_{A(B(m)+1)}$  with  $m \in \{1, 2, \dots, p(n)\}$ .

The representation of 0 which does not need any Fibonacci number is included in order to cover the case  $\ell = 0, k = 0$ .

Another set of generalized Chebyshev  $S_n$  polynomials is of interest. They are defined recursively by

$$(1.11a) \quad \hat{S}_{-1} = 0, \hat{S}_0 = 1,$$

$$(1.11b) \quad \hat{S}_n = Y(n+1)\hat{S}_{n-1} - \hat{S}_{n-2}, \quad n \geq 1,$$

with  $Y(n)$  defined by (1.1c). Table 2 shows  $\hat{S}_n(Y, y)$  for  $n = 0(1)13$ . They are given as  $(+i)^n$  times the denominator of the  $n^{\text{th}}$  approximation of the regular continued fraction

$$(1.12) \quad [0; -iY(2), -iY(3), \dots, -iY(k), \dots].$$

As far as combinatorics is concerned, one has to replace the numbers  $1, 2, \dots, n$  in the above given statements by the numbers  $2, 3, \dots, n + 1$ .

The physical motivation for considering the polynomials  $S_n(Y, y)$  and  $\hat{S}_n(Y, y)$  is sketched in the Appendix, where a set of  $2 \times 2$  matrices  $M_n$  formed from these polynomials is also introduced. In [14], [6], and [15],  $n$ -variable generalizations of Chebyshev's polynomials were introduced. For the 2-variable case, these polynomials satisfy a 4-term recursion formula and bear no relation to the ones studied in this work.

**Table 2.**  $\hat{S}_n = Y(n+1)\hat{S}_{n-1} - \hat{S}_{n-2}$ ,  $\hat{S}_{-1} = 0$ ,  $\hat{S}_0 = 1$   
 $Y(n+1) = Yh(n+1) + y(1 - h(n+1))$   
 $h(n+1) = [(n+2)\phi] - [(n+1)\phi] - 1$

$n$	$\hat{S}_n(Y, y)$
0	1
1	$y$
2	$Yy - 1 = S_2(Y, y)$
3	$Y^2y - (Y + y)$
4	$Y^2y^2 - y(2Y + y) + 1$
5	$S_5(Y, y)$
6	$Y^3y^3 - Yy^2(4Y + y) + 2y(2Y + y) - 1$
7	$Y^4y^3 - Y^2y^2(5Y + y) + Yy(7Y + 3y) - (3Y + y) = S_7(Y, y) - (Y - y)$
8	$Y^5y^3 - Y^3y^2(5Y + 2y) + Yy(7Y^2 + 7Yy + y^2) - (3Y^2 + 5Yy + 2y^2) + 1$
9	$Y^5y^4 - Y^3y^3(6Y + 2y) + Yy^2(12Y^2 + 8Yy + y^2) - y(10Y^2 + 8Yy + 2y^2) + (3Y + 2y)$
10	$S_{10}(Y, y)$
11	$Y^7y^4 - Y^5y^3(7Y + 3y) + Y^3y^2(17Y^2 + 16Yy + 3y^2) - Yy(17Y^3 + 27Y^2y + 11Yy^2 + y^3) + (6Y^3 + 17Y^2y + 10Yy^2 + 2y^3) - (4Y + 2y)$
12	$Y^7y^5 - Y^5y^4(8Y + 3y) + Y^3y^3(24Y^2 + 18Yy + 3y^2) - Yy^2(34Y^3 + 37Y^2y + 12Yy^2 + y^3) + y(23Y^3 + 32Y^2y + 13Yy^2 + 2y^3) - (6Y^2 + 11Yy + 4y^2) + 1$
13	$S_{13}(Y, y) + (Y - y)$
$\vdots$	

### 2. Fundamentals of Wythoff's Sequences

(see [22], [7], [21], [12], [8], [11], [9], [4], [19])

In this section we collect, without proofs, some well-known facts concerning Wythoff's pairs of natural numbers, the sequence  $\{h(n)\}$ , and their relation to the Fibonacci number system (1.9). We also introduce the counting sequences  $\{z(n)\}$  and  $\{p(n)\}$ .

The special Beatty sequences  $\{A(n)\}$  and  $\{B(n)\}$  (see [22], [9], [4]) given by (1.10) divide the set of natural numbers into two disjoint and exhaustive sets, henceforth called  $A$ - and  $B$ -numbers. For  $n = 0$  we also define the Wythoff pair  $(A(0), B(0)) = (0, 0)$ . The sequence  $h$ , defined in (1.2) as

$$(2.1) \quad h(n) = A(n+1) - A(n) - 1,$$

takes on values 0 and 1 only. Wythoff's pairs  $(A(n), B(n))$  have a simple characterization in the Fibonacci number system:  $A(n)$  is represented for each  $n \in \mathbb{N}$  with an *even* number of zeros at the end (including the case of no zero).  $B(n)$  is then obtained from the represented  $A(n)$  by inserting a 0 before the dot at the end. Therefore,  $B$ -numbers end in an *odd* number of zeros in this canonical number system. It is also known how to obtain the representation of  $A(n)$  from the given one for  $n$ .

The sequence  $h(n)$  (2.1) distinguishes the two types of numbers:

$$(2.2) \quad h(n) = \begin{cases} 0 & \text{iff } n \text{ is a } B\text{-number,} \\ 1 & \text{iff } n \text{ is an } A\text{-number.} \end{cases}$$

An  $A$ -number ending in a 1 in the Fibonacci system (no end zeros) has fractional part from the interval  $(2 - \phi, 2(2 - \phi))$ . Its fractional part is from the interval  $(2(2 - \phi), 1)$  if the  $A$ -number representation ends in at least two zeros. This distinction of  $A$ -numbers corresponds to the compositions

$$A(A(n)) \equiv A^2(n) = [[n\phi]\phi] \quad \text{and} \quad AB(n) = [[n\phi^2]\phi],$$

respectively.

It is convenient to introduce the projectors

$$(2.3) \quad k(n) := h(n) - (1 - h(n + 1)) = h(n)h(n + 1), \\ 1 - k(n) = (1 - h(n)) + (1 - h(n + 1)),$$

$k$  marks  $AB$ -numbers:

$$(2.4) \quad k(n) = \begin{cases} 1 & \text{iff } n \text{ is an } AB\text{-number,} \\ 0 & \text{otherwise.} \end{cases}$$

$A(B(m) + 1) = AB(m) + 1$ , i.e.,  $AB(m)$  is followed by an  $A$ -number. Such pairs of consecutive numbers will be called  $A$ -pairs. Some identities for  $n \in \mathbb{IN}$  which will be of use later on are:

$$(2.5a) \quad AB(n) = A(n) + B(n) = 2A(n) + n = B(A(n) + 1) - 2, \\ (2.5b) \quad BA(n) = 2A(n) + n - 1 = AB(n) - 1 = A(B(n) + 1) - 2, \\ (2.5c) \quad AA(n) = A(n) + n - 1 = B(n) - 1 = A(A(n) + 1) - 2, \\ (2.5d) \quad BB(n) = 3A(n) + 2n = ABA(n) + 2 = B(B(n) + 1) - 2, \\ = AAB(n) + 1.$$

No three consecutive numbers can be  $A$ -numbers, and no two consecutive numbers can be  $B$ -numbers. Among the  $AA$ -numbers  $\neq 1$ , we distinguish between those which are bigger members of an  $A$ -pair, viz,

$$(2.6) \quad AB(m) + 1 = A(B(m) + 1) = AA(A(m) + 1) \quad \text{for } m \in \mathbb{IN},$$

and the remaining ones which are called  $A$ -singles, viz,

$$(2.7) \quad AA(B(m) + 1) = A(AB(m) + 1) = BB(m) + 1 \quad \text{for } m \in \mathbb{IN}.$$

Thus,  $A$ -singles are  $AA$ -numbers having  $B$ -numbers as neighbors.  $A(n)$  is an  $A$ -single if  $h(n - 1) = h(n) = 1$ . The  $AA$ -number 1 is considered separately because we can either count  $(0, 1)$  as an  $A$ -pair or as a  $(B, A)$ -pair.

Define  $z(n)$  to be the number of (positive)  $A$ -numbers not exceeding  $n$ . This is

$$(2.8) \quad z(n) = \sum_{k=1}^n h(k) = [(n + 1)/\phi] = A(n + 1) - (n + 1).$$

The number of  $B$ -numbers  $\neq 0$  not exceeding  $n$  is then  $n - z(n) = [(n + 1)/\phi^2]$ .

Define  $p(n)$  to be the number of  $AB$ -numbers ( $0$  excluded) not exceeding  $n - 1$ . This is

$$(2.9) \quad p(n) = \sum_{m=1}^{n-1} k(m) = z(n) + z(n - 1) - n = 2A(n) - 3n + h(n).$$

The following identities hold:

$$(2.10) \quad pA(n + 1) = -A(n + 1) + 2n + 1 = n - z(n) = z^2(n - 1).$$

This is just the number of  $B$ -numbers (excluding 0) not exceeding  $n$ . The last equality follows with the help of

$$(2.11) \quad A(z(n-1) + 1) = A(A(n) - n + 1) = n + 1 - h(n),$$

which can be verified for  $A$ - and  $B$ -numbers  $n$  separately. Also,

$$(2.12) \quad pB(n) = A(n) - n = z(n-1),$$

$$(2.13) \quad pAB(m) = pBA(m) = m - 1.$$

The  $p$ -value increases by one at each argument  $AB(m) + 1$ , due to

$$(2.14) \quad k(n) = p(n+1) - p(n).$$

The  $p$ -value  $m$  appears  $2h(m) + 3$  times.

Another identity is

$$(2.15) \quad p(B(m) - 1) = pA^2(m) = z(m-1).$$

The number of  $A$ -singles ( $\neq 1$ ) not exceeding  $n$  is

$$(2.16) \quad pA(z(n) - p(n+1)) = pAz^2(n) = pz(n).$$

Finally,

$$(2.17) \quad z(n - z(n) - 1) = z(pA(n+1) - 1) = z(z^2(n-1) - 1) = p(n-1).$$

The last equality can be established by calculating  $B(n - z(n))$ .

$$(2.18) \quad \begin{aligned} B(n - z(n)) &= n + 1 - 2h(n) - h(n-1) \\ &= n - z(n) + z(n-2) + (1 - h(n)) = n - h(n) - k(n-1), \end{aligned}$$

implying

$$(2.19) \quad A(n - z(n)) = z(n) + 1 - 2h(n) - h(n-1) = n - z(n) + p(n-1).$$

### 3. Generalized Chebyshev Polynomials

Consider the recursion formula (1.1) with  $h(n)$  given by (1.2). For  $Y = y$ , the one for Chebyshev's  $S_n(y) \equiv S_n(y, y)$  polynomials [1] is found.\* Their explicit form is

$$(3.1) \quad S_n(y) = \sum_{\ell=0}^{\lfloor n/2 \rfloor} (-1)^\ell \binom{n-\ell}{\ell} y^{n-2\ell}, \quad n \in \mathbb{N}_0.$$

The binomial coefficient has, for  $\ell \neq 0$ , the following combinatorial meaning. It gives the number of ways to choose, from the numbers 1, 2, ...,  $n$ ,  $\ell$  mutually disjoint pairs of consecutive numbers. For  $\ell = 0$ , this number is put to 1. The sum over the moduli of the coefficients in (3.1), i.e., the sum over the "diagonals" of Pascal's triangle, is  $F_{n+1}$ . One also has

$$S_n(2) = n + 1 \quad \text{and} \quad S_n(3) = F_{2(n+1)},$$

which is proved by induction.

For  $Y \neq y$ , a certain two-variable generalization of these  $S_n$  polynomials results. We claim that they are given by (1.4) where the new coefficients have the combinatorial meaning given in Propositions 1-3 and the Corollary of the first section.

**Theorem 1:**  $S_n(Y, y)$  given by (1.4) with (1.5) and (1.6) is the solution of recursion formula (1.1) with (1.2) inserted.

\* $S_n(y) = U_n(y/2)$  with  $U_n(\cos \theta) = \sin((n+1)\theta)/\sin \theta$ , Chebyshev's polynomials of the second kind, for  $|y| < 2$ .

*Proof:* By induction over  $n$ . For  $n = 0$ ,  $k_{\min}(0, 0) = k_{\max}(0, 0) = 0$  due to  $z(0) = 0$  and, therefore,  $S_0 = 1$ . In order to compute  $S_m$  via (1.1), assuming (1.4) to hold for  $n = m - 1$  and  $n = m - 2$ , one writes

$$Y(m) = Y^{h(m)}y^{1-h(m)},$$

which is identical to (1.1c) due to the projector properties of the exponents. Now

$$z(m - 1) = z(m) - h(m) \quad \text{and} \quad z(m - 2) = z(m) - h(m) - h(m - 1),$$

following from (2.8) and (2.1), are employed to rewrite the  $Y$  and  $y$  exponents in the  $S_{m-1}$  term of (1.1b) such that exponents appropriate for  $S_m$  appear. In the  $S_{m-2}$  term of (1.1b) a factor  $(1/Y)^{k(m-1)}(1/y)^{1-k(m-1)}$  is in excess, which, when rewritten as  $k(m-1)(1/Y) + (1 - k(m-1))(1/y)$ , produces two terms from this  $S_{m-2}$  piece. In both of them the index shift  $\ell \rightarrow \ell - 1$  is performed, and in the first term  $k \rightarrow k - 1$  is used. Finally, one proves that the  $\ell$  and  $k$  range in all of the three terms which originated from  $S_{m-1}$  and  $S_{m-2}$  in (1.1b) can be extended to the one appropriate for  $S_m$  as claimed in (1.4). In order to show this, the convention to put  $(n; \ell, k)$  to zero as soon as for given  $n$  the indices  $\ell$  or  $k$  are out of the allowed range has to be followed. Also,

$$p(m - 2) = p(m) - k(m - 1) - k(m - 2),$$

resulting from (2.9), is used in the first term of  $S_{m-2}$  to verify that

$$k_{\max}(m - 2, \ell - 1) + 1 = k_{\max}(m, \ell).$$

In this term,  $m - 1$  is always an  $AB$ -number, and

$$k_{\min}(m - 2, \ell - 1) + 1 \geq k_{\min}(m, \ell)$$

holds as well. In the second term, which originated from  $S_{m-2}$ ,  $m - 1$  is not an  $AB$ -number, and one can prove that

$$k_{\min}(m - 2, \ell - 1) = k_{\min}(m, \ell) \quad \text{and} \quad k_{\max}(m - 2, \ell - 1) \leq k_{\max}(m, \ell).$$

In the  $S_{m-1}$  term one has, for even  $m$ , first to extend the upper  $\ell$  range by one, then the  $k$  range is extended as well, using

$$k_{\min}(m - 1, \ell) \geq k_{\min}(m, \ell) \quad \text{and} \quad k_{\max}(m - 1, \ell) \leq k_{\max}(m, \ell).$$

The coefficients of the three terms can now be combined under one  $k$ -sum and are just given by  $(m; \ell, k)$  due to recursion formula (1.5), which completes the induction proof. Our interest is now in the combinatorial meaning of the  $(n; \ell, k)$  defined by (1.5) with appropriate inputs.

*Lemma 1:*  $S_k$  defined by recursions (1.1a-c) satisfies, for  $k \in \mathbb{N}$ ,

$$(3.2) \quad S_k = Y(k) \dots Y(1) - Y(k) \dots Y(3)S_0 - Y(k) \dots Y(4)S_1 - \dots - Y(k)S_{k-3} - S_{k-2}.$$

*Proof:* By induction over  $k = 1, 2, \dots$ .

*Remark:* In (3.2) each of the  $k - 1$  terms with a minus sign can be obtained from the first reference term by deletion of one pair of consecutive

$$Y(i + 1)Y(i) \quad \text{for } i \in \{1, 2, \dots, k - 1\}$$

and by replacement of all  $Y(i - 1) \dots Y(1)$  following to the right by  $S_{i-1}$ . So there is a one-to-one correspondence between these  $k - 1$  terms and the  $k - 1$  different pairs of consecutive numbers that can be picked out of  $\{1, 2, \dots, n\}$ .

*Notation:* The  $k - 1$  terms of  $S_k - Y(k) \dots Y(1)$  given by (3.2) are denoted by  $[i, i + 1]$ , with  $i = 1, 2, \dots, k - 1$ . E.g., for  $k = 5$ ,  $[3, 4] \equiv -Y(5)S_2$ , i.e.,  $Y(4)$  and  $Y(3)$  do not appear.

*Lemma 2:*  $S_k$  of (3.2) consists in all of  $F_{k+1}$  terms if all  $S_i$  appearing on the right-hand side of (3.2) are iteratively inserted until only products of  $Y$ 's occur.

*Proof:* By induction, using  $S_0 = 1$  and  $1 + \sum_{i=1}^{k-1} F_i = F_{k+1}$ .

*Definition 1:*  $Q(n)$  is the set of  $F_{n+1} - 1$  elements given by the individual terms of which  $S_n - Y(n) \dots Y(1)$  consists due to Lemma 2.

*Definition 2:*  $P_\ell(n)$ , for  $\ell \in \{1, 2, \dots, [n/2]\}$ , is the set of  $\ell$  mutually disjoint pairs of consecutive numbers taken out of the set  $\{1, 2, \dots, n\}$ .

*Lemma 3:* The elements of  $Q(n)$  are given by

$$q_{\ell, i}(n) := (-1)^\ell Y(n) \dots \overline{Y(n_{i_\ell} + 1) \cdot Y(n_{i_\ell})} \dots \overline{Y(n_{i_1} + 1) \cdot Y(n_{i_1})} \dots Y(1),$$

where the  $\ell$  barred  $Y$ -pairs have to be omitted and

$$(n_{i_1}, n_{i_1} + 1), \dots, (n_{i_\ell}, n_{i_\ell} + 1)$$

is an element of  $P_\ell(n)$  for  $\ell = 1, 2, \dots, [n/2]$ . The index  $i$  numerates the different  $\ell$  pairs:

$$i = 1, 2, \dots, \binom{n - \ell}{\ell}.$$

*Proof:* Let  $(n_1, n_1 + 1), \dots, (n_\ell, n_\ell + 1)$  with  $n_j > n_{j-1} + 1$  for  $j = 2, \dots, \ell$  be an element of  $P_\ell(n)$ . Using the Notation, the corresponding element of  $Q(n)$  is obtained by picking in the  $[n_\ell, n_\ell + 1]$  term of  $S_n$  the  $[n_{\ell-1}, n_{\ell-1} + 1]$  term of  $S_{n_\ell-1}$  which appears there, and so on, until the  $[n_1, n_1 + 1]$  term of  $S_{n_2-1}$  is reached. If  $n_1 = 1$ , one arrives at  $S_0 = 1$ . If  $n_1 \geq 2$ , one replaces the surviving  $S_{n_1-1}$  by its first term, i.e.,  $Y(n_1 - 1) \dots Y(1)$ . In this way, each of the  $\binom{n-\ell}{\ell}$  elements of  $P_\ell(n)$ , distinguished by the label  $i$ , is mapped to a different element of  $Q(n)$ . For all  $\ell$ , there are in all  $F_{n+1} - 1$  such elements, and this mapping from  $\cup_{\ell=1}^{[n/2]} P_\ell$  to  $Q(n)$  is one-to-one. It is convenient also to define  $q_0 := Y(n) \dots Y(1)$ , which is the first term of  $S_n$ .

*Lemma 4:* (3.3)  $q_0 = Y^{z(n)} y^{n-z(n)}$ .

*Proof:* Definition (2.9) of counting sequence  $z(n)$ .

*Lemma 5:* The general element  $q_{\ell, i}(n) \in Q(n)$  is given by

$$(3.4) \quad q_{\ell, i}(n) = Y^{z(n)} y^{n-z(n)} \{ (-1)^\ell Y^{-(2k+\ell-k)} y^{-(\ell-k)} \},$$

if among the specific choice  $i$  of  $\ell$  barred pairs of  $q_{\ell, i}(n)$ , as written in Lemma 3,  $k$  barred pairs are numerated by  $A$ -numbers.

*Proof:* A barred pair  $Y(i + 1)Y(i)$  in  $q_{\ell, i}(n)$ , given in Lemma 3, corresponds to a missing factor  $-Y^2$  in  $Y(n) \dots Y(1)$  iff  $i$  and  $i + 1$  are both  $A$ -numbers. In all other cases a factor  $-Yy$  is missing. Therefore, the reference term  $q_0$  of (3.3) is changed as stated in (3.4).

Putting these results together, we have proved Proposition 2 given in the first section, because the elements of  $Q(n) \cup q_0$  are all the terms of  $S_n$ , and the multiplicity of a term with fixed powers of  $Y$  and  $y$  given in (3.4) is just  $(n; \ell, k)$  according to (1.4).

Proposition 1 is equivalent to Proposition 2 because of the characterization of  $A$ -numbers in the Fibonacci number system, as described in section 2.



If a pair of consecutive numbers is replaced by its smaller member, Proposition 3 results from either Proposition.

The Corollary follows from Proposition 3 and the Fibonacci representation explained in (1.9). The numbers  $1, 2, \dots, n - 1$  indicate the places  $F_2, F_3, \dots, F_n$ , respectively. In (1.9)  $s_{i-1} = 1$  if the number  $i \in \{1, 2, \dots, n - 1\}$  is chosen. If  $i = AB(m)$ , for some  $m \in \mathbb{N}$ , the place of

$$F_{AB(m)+1} = F_{A(B(m)+1)}$$

is activated.

*Comment:* The map used in the proof of Lemma 3 never produces negative powers of  $Y$  or  $y$ . Thus,

$$\ell - (n - z(n)) \leq k \leq z(n) - \ell$$

is always obeyed. On the other hand, the  $p(n)$  definition shows that

$$0 \leq k \leq \min(\ell, p(n))$$

has to hold as well. (1.6) gives the intersection of both  $k$  ranges.

The main part of this work closes with a collection of explicit formulas concerning the  $(n; \ell, k)$  numbers. Here, the results listed in section 2 are used.

A necessary condition is

$$(3.5) \quad \sum_{k=k_{\min}}^{k_{\max}} (n; \ell, k) = \binom{n - \ell}{\ell},$$

which guarantees  $S_n(y, y) = S_n(y)$ .

The results for  $(n; \ell, k)$  for  $\ell = 0, 1, 2$ , are:

$$(3.6) \quad \underline{\ell = 0}: (n; 0, 0) = 1,$$

$$(3.7a) \quad \underline{\ell = 1}: (n; 1, 0) = (n - 1) - p(n),$$

$$(3.7b) \quad (n; 1, 1) = p(n),$$

$$(3.8a) \quad \underline{\ell = 2}: (n; 2, 0) = \binom{p(n)}{2} + p(n - 1) - (n - 3)p(n) + \binom{n - 2}{2},$$

$$(3.8b) \quad (n; 2, 1) = (n - 3)p(n) - p(n - 1) - 2\binom{p(n)}{2},$$

$$(3.8c) \quad (n; 2, 2) = \binom{p(n)}{2}.$$

Already the  $\ell = 3$  case becomes quite involved, except for  $(n; 3, 3)$ , which is a special case of

$$(3.9) \quad (n; \ell, \ell) = \binom{p(n)}{\ell}, \quad \text{for } n \geq AB(\ell) + 1.$$

This is, from the combinatorial point of view, a trivial formula, which, when derived from the recursion formula, is due to an iterative solution of

$$(n; \ell, \ell) = \sum_{k=0}^{p(n)} (BA(k); \ell - 1, \ell - 1),$$

with input  $(BA(k); 0, 0) = 1$ .

The last term of  $S_{2\ell}$  has just the coefficient

$$(3.10) \quad (2\ell; \ell, z(2\ell) - \ell) = 1,$$

where the input  $(2; 1, 0) = 1$  was used.

Finally, we list some questions that are under investigation:

- (i) What do the generating functions for  $S_n$ ,  $\hat{S}_n$  look like?
- (ii) Which differential equations do these objects satisfy?
- (iii) Are the  $S_n$  and  $\hat{S}_n$  orthogonal with respect to some measure?
- (iv) How does the self-similarity of the  $h(n)$  sequence reflect itself in the polynomials  $S_n$  and  $\hat{S}_n$ ?

## APPENDIX

### Physical Applications

The two-variable polynomials introduced in this work are basic for the solution of the discrete one-dimensional Schrödinger equation for a particle of mass  $m$  moving in a quasi-periodic potential of the Fibonacci type (see [13] and [17]). The transfer matrix for such a model is given by

$$(A.1) \quad R_n := \begin{pmatrix} Y(n), & -1 \\ 1, & 0 \end{pmatrix},$$

with  $Y(n)$  defined by (1.1c) and (1.2).  $Y = E - V_1$ ,  $y = E - V_0$ , where  $E$  is the energy (in units of  $\hbar^2/2ma^2$ , with lattice constant  $a$ ) and the potential at lattice site  $n$  is  $V_n := V(n\phi)$  with

$$(A.2) \quad V(x) = \begin{cases} V_0 & \text{for } 0 \leq x < 2 - \phi \\ V_1 & \text{for } 2 - \phi \leq x < 1 \end{cases} \quad \text{and } V(x+1) = V(x).$$

The product matrix

$$(A.3) \quad M_n := R_n \cdots R_2 R_1,$$

which allows us to compute  $\psi_n$ , the particle's wave-function at site number  $n$ , in terms of the inputs  $\psi_1$  and  $\psi_0$ , according to

$$(A.4) \quad \begin{pmatrix} \psi_{n+1} \\ \psi_n \end{pmatrix} = M_n \begin{pmatrix} \psi_1 \\ \psi_0 \end{pmatrix}$$

turns out to be

$$(A.5) \quad M_n = \begin{pmatrix} S_n, & -\hat{S}_{n-1} \\ S_{n-1}, & -\hat{S}_{n-2} \end{pmatrix}.$$

Because of  $\det R_n = 1 = \det M_n$ , one finds the identity

$$(A.6) \quad \hat{S}_n S_n - \hat{S}_{n-1} S_{n+1} = 1,$$

for  $n \in \mathbb{N}$ , which generalizes a well-known result for ordinary Chebyshev polynomials. It allows to express  $\hat{S}_n$  in terms of  $S_i$  with  $i = 0, 1, \dots, n+1$ :

$$(A.7) \quad \hat{S}_n = \frac{1}{S_n} \left( 1 + S_n S_{n+1} \sum_{i=0}^{n-1} \frac{1}{S_i S_{i+1}} \right),$$

This can be proved by induction using

$$\hat{S}_n = \frac{1}{S_n} (1 + S_{n+1} \hat{S}_{n-1}).$$

Another model that leads to the same type of transfer matrices as (A.1) is the Fibonacci chain [2] with harmonic nearest neighbor interaction built from two masses  $m_0$  and  $m_1$  with mass  $m_{h(i)}$  at site number  $i$ . In this case

$$Y(n) = 2 - (\omega/\omega(n))^2, \text{ with } \omega^2(n) := \kappa/m_{h(n)}.$$

$\kappa$  is the spring constant and  $\omega$  the frequency.

One-dimensional quasi-crystal models (see [16], [3]) can be transformed to Schrödinger equations on a regular lattice with quasi-periodic potentials as considered above.

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