ANOTHER GENERALIZATION OF GOULD'S STAR OF DAVID THEOREM

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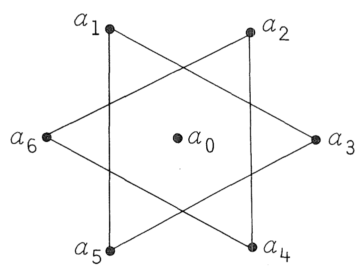
## 1. Introduction

Let $a_1$, $a_2$, $a_3$, $a_4$, $a_5$, and $a_6$ denote the hexagon of elements immediately surrounding any given element $a_0$ in Pascal's triangle.




Since the first paper by Hoggatt & Hansell [8] showing that $a_1a_3a_5 = a_2a_4a_6$ and hence that $\prod_{i=1}^{6} a_i = k^2$ for some integer $k$, a number of papers examining the properties of these arrays and their generalizations have appeared. Among the more surprising of these is the GCD Star of David theorem that

$$(a_1,\ a_3,\ a_5) = (a_2,\ a_4,\ a_6)$$

conjectured by Gould [4] and proved and/or generalized by Hillman & Hoggatt [5] and [6], Strauss [11], Singmaster [10], Hitotumatu & Sato [7], Ando & Sato [1], [2], and [3], and Long & Ando [9]. In the last listed paper, it was shown that

$$(a_1,\ a_3,\ \ldots,\ a_{17}) = (a_2,\ a_4,\ \ldots,\ a_{18})$$

where the $a$ , $1 \le i \le 18$, are the eighteen adjacent binomial coefficients in the regular hexagon of coefficients centered on any particular coefficient $\binom{n}{r}$ and that

$$(b_1,\ b_3,\ \ldots,\ b_{11}) = t \cdot (b_2,\ b_4,\ \ldots,\ b_{12})$$

where the $b$ , $1 \le i \le 12$, are the twelve adjacent binomial coefficients in the regular hexagon of coefficients centered at $\binom{n}{r}$ with $t = 1$ if $r$ or $n - r = s$ is even, $t = 2$ if $r$ and $s$ are odd and $r \equiv 3 \pmod 4$ or $s \equiv 3 \pmod 4$, and $t = 4$ if $r \equiv s \equiv 1 \pmod 4$. Moreover, it was conjectured that

$$(a_1,\ a_3,\ \ldots,\ a_{2m-1}) = (a_2,\ a_4,\ \ldots,\ a_{2m})$$

if the $a_i$, $1 \le i \le 2m$, are the coefficients in a regular hexagon of binomial coefficients with edges along the rows and main diagonals of Pascal's triangle and with an even number of coefficients per edge. For such regular hexagons but with an odd number of coefficients per edge it was conjectured that

$$(a_1,\ a_3,\ \ldots,\ a_{2m-1}) = t \cdot (a_2,\ a_4,\ \ldots,\ a_{2m})$$

where $t$ is a "simple" rational number depending on $m$, $n$, and $r$. In the present paper, we show that the regularity condition on the hexagons with an even number of coefficients per side is not necessary. In fact, we now conjecture that the equal gcd property holds for convex hexagons of adjacent entries along the rows and main diagonals of Pascal's triangle provided there are $2u$, $2v$, $2w$, $2u$, $2v$, and $2w$ coefficients on the consecutive sides. Being unable to prove the conjecture in general, we here prove it for the case $u = 3$, $v = 2$, and $w = 1$.

## 2. Some Preliminaries

Throughout the paper small Latin letters will always denote integers. Let $r + s = n$ as above, set $A = \binom{n}{r}$ and, for simplicity, set

$$(h,\ k) = \binom{n + h + k}{r + h}.$$

Let $p$ be a prime. For any rational number $\alpha$, there exists a unique integer $v = v(\alpha)$ such that $\alpha = p^v a/b$ where $(a,\ p) = (b,\ p) = 1$. If $v(n) = e$, then $p^e \| n$; i.e., $p^e | n$ and $p^{e+1} \nmid n$. Moreover, it is clear that

$$v(1) = 0, \tag{1}$$
$$v(\alpha\beta) = v(\alpha) + v(\beta), \tag{2}$$
$$v(\alpha/\beta) = v(\alpha) - v(\beta), \tag{3}$$
$$v(\alpha \pm \beta) \geq \min(v(\alpha),\ v(\beta)) \quad \forall \alpha,\ \beta, \tag{4}$$
$$v(\alpha \pm \beta) = \min(v(\alpha),\ v(\beta)) \quad \text{if } v(\alpha) \neq v(\beta). \tag{5}$$

Finally, if $m = m_1 m_2 \ldots m_k$, then

$$(m_1,\ m_2,\ \ldots,\ m_k) = \prod_{p|m} p^{\min(v(m_1),\ \ldots,\ v(m_k))}. \tag{6}$$

## 3. The Main Result

Now consider the eighteen binomial coefficients forming a hexagon centered at $A$ as indicated in Figure 1. Let

$$S_1 = \{a_1,\ a_3,\ \ldots,\ a_{17}\},\quad S_2 = \{a_2,\ a_4,\ \ldots,\ a_{18}\},$$
$$\gcd S_1 = (a_1,\ a_3,\ \ldots,\ a_{17}),\quad \gcd S_2 = (a_2,\ a_4,\ \ldots,\ a_{18}).$$

Then, using the notation $(h,\ k)$ above,

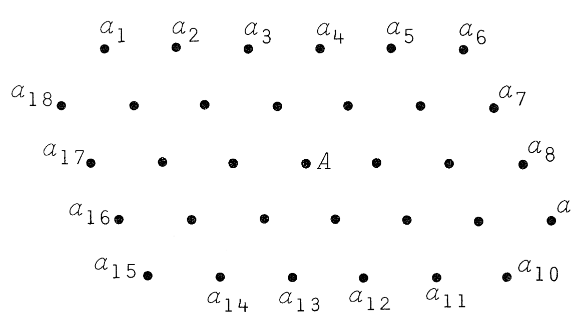



**Figure 1**

we can list the elements of $S_1$ and $S_2$ as in Table 1.

It is clear from the table that the product of the elements in $S_1$ is equal to the product of those in $S_2$ and it is not difficult to show by counter example that lcm $S_1$ = lcm $S_2$ is not always true. In particular, if $A = \binom{11}{5}$,

$$\text{lcm } S_1 = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \quad \text{and} \quad \text{lcm } S_2 = 2^2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13,$$

so lcm $S_1 \neq$ lcm $S_2$. However, the result shown in the Theorem below does hold.

Table 1

| $S_1 = S_1(n, r)$ | $S_2 = S_2(n, r)$ |
|---|---|
| $(-4, 3) = \frac{r(r-1)(r-2)(r-3)}{n(s+1)(s+2)(s+3)}A$ | $(-4, 2) = \frac{r(r-1)(r-2)(r-3)}{n(n-1)(s+1)(s+2)}A$ |
| $(-2, 3) = \frac{r(r-1)(r+1)}{(s+1)(s+2)(s+3)}A$ | $(-3, 3) = \frac{r(r-1)(r-2)}{(s+1)(s+2)(s+3)}A$ |
| $(0, 2) = \frac{(n+1)(n+2)}{(s+1)(s+2)}A$ | $(-1, 3) = \frac{r(n+1)(n+2)}{(s+1)(s+2)(s+3)}A$ |
| $(2, 0) = \frac{(n+1)(n+2)}{(r+1)(r+2)}A$ | $(1, 1) = \frac{(n+1)(n+2)}{(r+1)(s+1)}A$ |
| $(4, -2) = \frac{s(s-1)(n+1)(n+2)}{(r+1)(r+2)(r+3)(r+4)}A$ | $(3, -1) = \frac{s(n+1)(n+2)}{(r+1)(r+2)(r+3)}A$ |
| $(3, -3) = \frac{s(s-1)(s-2)}{(r+1)(r+2)(r+3)}A$ | $(4, -3) = \frac{s(s-1)(s-2)(n+1)}{(r+1)(r+2)(r+3)(r+4)}A$ |
| $(1, -3) = \frac{s(s-1)(s-2)}{n(n-1)(r+1)}A$ | $(2, -3) = \frac{s(s-1)(s-2)}{n(r+1)(r+2)}A$ |
| $(-1, -1) = \frac{rs}{n(n-1)}A$ | $(0, -2) = \frac{s(s-1)}{n(n-1)}A$ |
| $(-3, 1) = \frac{r(r-1)(r-2)}{n(n-1)(s+1)}A$ | $(-2, 0) = \frac{r(r-1)}{n(n-1)}A$ |

*Theorem:* For any $n \geq 7$, $r \geq 4$, $s \geq 4$, with $r + s = n$ and $S_1$ and $S_2$ as above, $\gcd S_1 = \gcd S_2$.

*Proof:* Let $p$ be any prime and, for convenience, set $v((a, b)) = v(a, b)$. Also, set

$$v_i = v_i(p) = \min_{(a, b) \in S_i} \{v(a, b)\}, \quad i = 1, 2.$$

Clearly, we must show that $v_1 = v_2$ for all $p$. In fact, we show that both assumptions $v_1 < v_2$ and $v_2 < v_1$ lead to contradictions, so the desired equality must hold. Actually, the proof is not elegant. Since we can use neither symmetry nor rotation arguments, it is necessary to consider individually the nine cases where we successively let $v_1 = v(a_i)$, $a_i \in S_1$, and show each time that the assumption $v_1 < v_2$ leads to a contradiction. It is also necessary to consider individually the nine cases where $v_2 = v(a_i)$, $a_i \in S_2$, and show each time that the assumption $v_2 < v_1$ leads to a contradiction. In fact, since all these arguments are very similar, we only prove case 1, where we take $v_1 = v(-4, 3) < v_2$.

For $(a, b) \in S_i$, let $u((a, b)) = u(a, b) = v(a, b) - v(A)$ and let $u_i = v_i - v(A)$ for each $i$. With this notation, it is clear that the assumption $v_1 < v_2$ is equivalent to $u_1 < u_2$. First, assume that $p$ is odd. The assumption $u_1 < u_2$ implies that $u_1 < u(a_i)$ for all $a_i \in S_2$. Therefore, in particular,

$$u_1 < u(-4, 2) \quad \text{and} \quad u_1 < u(-3, 3);$$

that is,

$$v\left(\frac{r(r-1)(r-2)(r-3)}{n(s+1)(s+2)(s+3)}\right) < v\left(\frac{r(r-1)(r-2)(r-3)}{n(n-1)(s+1)(s+2)}\right) \tag{7}$$

and

$$v\left(\frac{r(r-1)(r-2)(r-3)}{n(s+1)(s+2)(s+3)}\right) < v\left(\frac{r(r-1)(r-2)}{(s+1)(s+2)(s+3)}\right). \tag{8}$$

But, using (5), (7), and (8) clearly implies that

$$v(s+3) > v(n-1) = v(r-4) \geq 0 \tag{9}$$

and

(10) $\quad v(n) > v(r - 3) = v(s + 3) > 0,$

whence it follows that $p|n$, $p|(s + 3)$, and $p|(r - 3)$ since $r + s = n$. But now, since $p$ is odd,

(11) $\quad p\nmid(n - 1)(r - 1)(r - 2)(s + 1)(s + 2)$

and it follows that

(12) $$u(-2, 0) = v\left(\frac{r(r - 1)}{n(n - 1)}\right) = v\left(\frac{r(r - 1)(r - 2)(r - 3)}{n(s + 1)(s + 2)(s + 3)}\right) = u_1$$

contrary to the assumption that $u_1 < u_2$, since $(-2, 0) \in S_2$.

Now assume that $p = 2$. Then all of the above up to, but not including (11), still holds and we may conclude that $n$ is even and $r$ and $s$ are odd. Thus, $2\nmid r(r - 2)(r + 2)s(s + 2)(n + 1)$. Also, $v(s + 3) > 0$ in (10); hence $v(n) \geq 2$. But this implies that $v(n + 2) = 1$ since every second even integer is divisible by only $2^1$ and no higher power. If $v(n) \leq v(r - 1)$, then $v(r - 1) \geq 2$ and

$$u(-1, 3) = v\left(\frac{r(n + 1)(n + 2)}{(s + 1)(s + 2)(s + 3)}\right) \leq v\left(\frac{r(r - 1)(r - 2)(r - 3)}{n(s + 1)(s + 2)(s + 3)}\right) = u_1$$

contrary to the assumption that $u_1 < u_2$ since $u(-1, 3) \in S_2$. Therefore, again using (5), $v(n) > v(r - 1) = v(s + 1)$. If $v(n) \leq v(r + 1)$, then

$$u(1, 1) = v\left(\frac{(n + 1)(n + 2)}{(r + 1)(s + 1)}\right) \leq v\left(\frac{r(r - 1)(r - 2)(r - 3)}{n(s + 1)(s + 2)(s + 3)}\right) = u_1$$

since $v(n + 2) = 1 \leq v(s + 1)$ from above. Since this is again a contradiction, it follows that $v(n) > v(r + 1) = v(s - 1)$ by (5). But then

$$u(2, -3) = v\left(\frac{s(s - 1)(s - 2)}{n(r + 1)(r + 2)}\right) = v\left(\frac{r(r - 1)(r - 2)(r - 3)}{n(s + 1)(s + 2)(s + 3)}\right) = u_1$$

by (10), and this again contradicts the assumption $u_1 < u_2$ since $u(2, -3) \in S_2$.

Since similar arguments lead to contradictions in all the remaining seventeen cases, we conclude that $v_1 = v_2$ for all $p$ and hence that gcd $S_1$ = gcd $S_2$ as claimed.

We note that this argument, as in the preceding paper [9], depends on the fact that we have only a very finite number of cases to consider. The general argument for hexagons of arbitrary size will have to be much different and much more sophisticated.

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