AREA-BISECTING POLYGONAL PATHS

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Introduction

For a rather striking geometrical property of the Fibonacci sequence

$$f_0 = f_1 = 1 \quad \text{and} \quad f_n = f_{n-1} + f_{n-2} \quad (n = 2, 3, \ldots),$$

consider the lattice points defined by $F_0 = (0, 0)$ and

$$F_n = (f_{n-1}, f_n), \quad X_n = (f_{n-1}, 0), \quad Y_n = (0, f_n) \quad (n = 1, 2, 3, \ldots).$$

Then, as we shall prove: for each $n \geq 1$, the polygonal path

$$F_0F_1F_2 \ldots F_{2n+1}$$

splits the rectangle

$$F_0X_{2n+1}F_{2n+1}Y_{2n+1}$$

into two regions of equal area. Figure 1 illustrates this area-splitting property for $n = 0, 1, 2$.

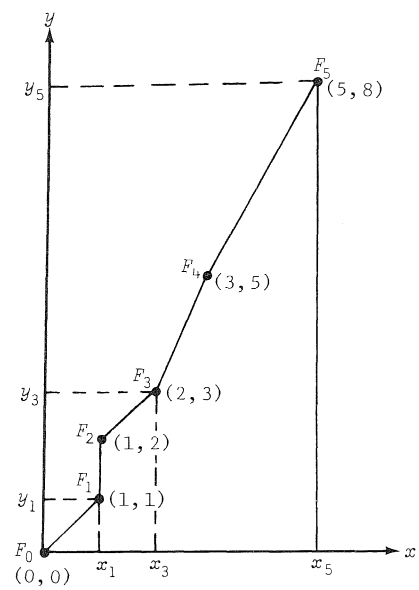

Figure 1

In view of the above, it seems only natural to ask if there exist other types of area-splitting paths, and how they may be characterized. To give some answers, it will be convenient to introduce the following notation and terminology.

We henceforth assume that every point with zero subscript is the origin. In particular, $P_0 = (0, 0)$ and each point $P_n = (x_n, y_n)$ has projections $X_n = (x_n, 0)$ and $Y_n = (0, y_n)$ on the axis. We shall also assume that a polygonal path has distinct vertices (that is, $P_n \neq P_m$ for $n \neq m$).

A polygonal path $P_0P_1P_2\ldots$ will be called *nondecreasing* if the abscissas and the ordinates of its vertices P_0, P_1, P_2, ... are each nondecreasing sequences. An *area-bisecting* k-path ($k \geq 2$) is a nondecreasing path $P_0P_1P_2\ldots$ that satisfies

$$\text{(1)} \qquad \text{Area}\{P_0P_1P_2\ldots P_{nk+1}X_{nk+1}\} = \text{Area}\{P_0P_1P_2\ldots P_{nk+1}Y_{nk+1}\}$$

for each integer $n \geq 0$.

An area-bisecting k-path is an area-bisecting Nk-path for each natural number N. The converse, however, is false. In Figure 2, any area-bisecting 4-path beginning with $P_0P_1P_2P_3P_4P_5$ cannot be an area-bisecting 2-path because area$\{X_1P_1P_2P_3X_3\}$ is not equal to area$\{Y_1P_1P_2P_3Y_3\}$.

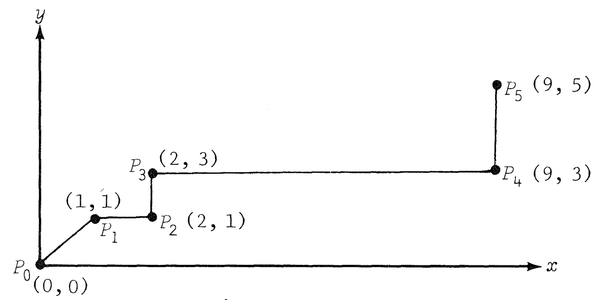

Figure 2

To characterize the situation when (1) holds, consider any segment $P_m P_{m+1}$ of the path $P_0P_1P_2\ldots$ (Fig. 3). Since

$$\text{(2a)} \qquad 2 \cdot \text{area}\{X_mP_mP_{m+1}X_{m+1}\} = x_{m+1}y_{m+1} - x_my_m - \begin{vmatrix} x_m & y_m \\ x_{m+1} & y_{m+1} \end{vmatrix}$$

and

$$\text{(2b)} \qquad 2 \cdot \text{area}\{Y_mP_mP_{m+1}Y_{m+1}\} = x_{m+1}y_{m+1} - x_my_m - \begin{vmatrix} x_m & y_m \\ x_{m+1} & y_{m+1} \end{vmatrix}$$

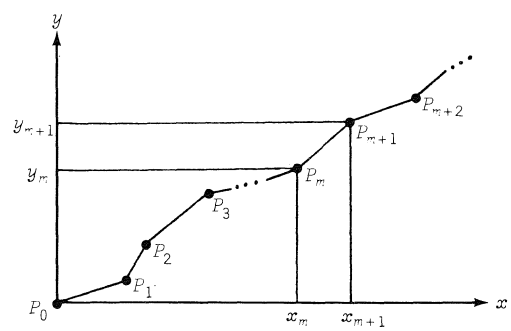

Figure 3

we see that (1) holds for each $n \geq 0$ if and only if the determinantal equation

$$\text{(3)} \qquad \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \cdots + \begin{vmatrix} x_{nk} & y_{nk} \\ x_{nk+1} & y_{nk+1} \end{vmatrix} = 0$$

holds for each $n \geq 1$. This can be summarized as follows.

Theorem 1: A nondecreasing path $P_0P_1P_2\ldots$ is an area-bisecting k-path if and only if

$$(4)\qquad \begin{vmatrix} x_{nk+1} & y_{nk+1} \\ x_{nk+2} & y_{nk+2} \end{vmatrix} + \begin{vmatrix} x_{nk+2} & y_{nk+2} \\ x_{nk+3} & y_{nk+3} \end{vmatrix} + \cdots + \begin{vmatrix} x_{(n+1)k} & y_{(n+1)k} \\ x_{(n+1)k+1} & y_{(n+1)k+1} \end{vmatrix} = 0$$

for each $n \geq 0$.

Remark: $P_0P_1P_2\ldots$ is an area-bisecting k-path if and only if its reflection about the line $y = x$ is an area-bisecting k-path.

To confirm that the Fibonacci path $F_0F_1F_2\ldots$ is area-bisecting, set $P_0 = F_0$ and let P_n be the point $F_n = (f_{n-1}, f_n)$ for each $n \geq 1$. For $k = 2$, condition (4) reduces to

$$\begin{vmatrix} f_{2n-2} & f_{2n-1} \\ f_{2n-1} & f_{2n} \end{vmatrix} + \begin{vmatrix} f_{2n-1} & f_{2n} \\ f_{2n} & f_{2n+1} \end{vmatrix} = 0$$

for each $n \geq 1$. This is clearly true since $f_i = f_{i-1} + f_{i-2}$ for each $i \geq 2$. Verification that $F_0F_1F_2\ldots$ is an area-bisecting 2-path can also be obtained by letting $\alpha = \beta = 1 = k - 1$ and setting $s_1 = f_0$ and $s_2 = f_1$ in the following.

Corollary 1.1: Let $S_n = (s_n, s_{n+1})$ for the positive sequence

$$(5)\qquad s_1,\ s_2,\ \text{and}\ s_n = \beta s_{n-1} + \alpha s_{n-2}\quad (n \geq 3).$$

Then $S_0S_1S_2\ldots$ is an area-bisecting k-path if and only if:

(i) k is even and $\alpha = 1$ for nondecreasing $\{s_n : n \geq 1\}$

or

(ii) $s_2^2 = s_1s_3$ [which is equivalent to $S_0S_1S_2\ldots$ being embedded in the straight line $y = (s_2/s_1)x$].

Proof: First, observe that $s_2^2 = s_1s_3$ yields

$$s_3^2 = (\beta s_2 + \alpha s_1)s_3 = \beta s_2 s_3 + \alpha s_2^2 = s_2 s_4$$

and (by induction)

$$(6)\qquad s_{n+1}^2 = s_n s_{n+2}\quad (n \geq 1).$$

Since this is equavalent to

$$(7)\qquad \frac{s_{n+1}}{s_n} = \frac{s_2}{s_1}\quad (n \geq 1),$$

$s_2^2 = s_1s_3$ is equivalent to $S_0S_1S_2\ldots$ being contained in the line $y = (s_2/s_1)x$.

Conditions (i) and (ii) each ensure that $S_0S_1S_2\ldots$ is nondecreasing. Moreover, by (4), this path is an area-bisecting k-path if and only if

$$(8)\qquad \begin{vmatrix} s_{nk+1} & s_{nk+2} \\ s_{nk+2} & s_{nk+3} \end{vmatrix} + \cdots + \begin{vmatrix} s_{(n+1)k-1} & s_{(n+1)k} \\ s_{(n+1)k} & s_{(n+1)k+1} \end{vmatrix} + \begin{vmatrix} s_{(n+1)k} & s_{(n+1)k+1} \\ s_{(n+1)k+1} & s_{(n+1)k+2} \end{vmatrix} = 0$$

for each $n \geq 0$. Now observe that for each $m \geq 2$,

$$(9)\qquad s_m s_{m+2} - s_{m+1}^2 = -\alpha(s_{m-1}s_{m+1} - s_m^2)$$

follows from $s_{m+2} = \beta s_{m+1} + \alpha s_m$ and $s_{m+1} = \beta s_m + \alpha s_{m-1}$. Therefore, using (9) in successively recasting each determinant in (8) beginning with the rightmost determinant, we find that (8) is equivalent to

$$(10)\qquad (1 - \alpha + \alpha^2 - \cdots + (-\alpha)^{k-1})(s_{nk+1}s_{nk+3} - s_{nk+2}^2) = 0$$

for each $n \geq 0$. In particular, (10) holds for all $n \geq 0$ if and only if

$$\sum_{t=0}^{k-1}(-\alpha)^t = 0 \quad \text{or} \quad s_2^2 = s_1 s_3.$$

For real values of α, it is easily verified that

$$\sum_{t=0}^{k-1}(-\alpha)^t = 0 \text{ if and only if } k \text{ is even and } \alpha = 1.$$

Matrix-Generated Paths

Since the Fibonacci numbers can be generated by powers of the matrix

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \left(\text{via } C_n = \begin{bmatrix} f_{n-2} & f_{n-1} \\ f_{n-1} & f_n \end{bmatrix} \text{ for each } n \geq 2\right),$$

the consecutive vertices $\{F_n = (f_{n-1}, f_n) : n = 1, 2, \ldots\}$ of the Fibonacci path are given precisely by the successive rows

$$\{F_{2n-1} = (f_{2n-2}, f_{2n-1}) \quad \text{and} \quad F_{2n} = (f_{2n-1}, f_{2n})\}$$

of $(C^2)^n$:

$$C^{2n} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^n = \begin{bmatrix} f_{2n-2} & f_{2n-1} \\ f_{2n-1} & f_{2n} \end{bmatrix} \quad (n \geq 1).$$

Thus, the Fibonacci path $F_0F_1F_2\ldots$ is generated by $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ in the following sense:

A path $P_0P_1P_2\ldots$ is said to be *matrix-generated* by $\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$ if

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}^n = \begin{bmatrix} x_{2n-1} & y_{2n-1} \\ x_{2n} & y_{2n} \end{bmatrix} \quad \text{for each } n \geq 1.$$

Example 1:

(i) If $S_n = (s_n, s_{n+1})$ for $(s_1, s_2) = (1, 2)$ and $s_n = s_{n-1} + 2s_{n-2}$, the area-bisecting path $S_0S_1S_2\ldots$ (contained in the line $y = 2x$) cannot be matrix-generated since the first row of

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}^2 = \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix}$$

is not $S_3 = (4, 8)$. Note, however, that $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ generates an area-bisecting path whose consecutive vertices are the successive rows of $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}^n$.

(ii) The area-bisecting path $P_0P_1P_2\ldots$ cannot be matrix-generated when $P_n = (f_{n-2}, f_{n-1})$ for the Fibonacci sequence beginning with $f_{-1} = 0$, or when $P_n = (\ell_n, \ell_{n+1})$ for the Lucas sequence beginning with $(\ell_1, \ell_2) = (1, 3)$ and $\ell_n = \ell_{n-1} + \ell_{n-2}$ $(n \geq 3)$.

(iii) The path in Figure 4 cannot be matrix-generated because

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} a & 0 \\ a & b \end{bmatrix}$$

is nonsingular, whereas points P_0, P_n, P_{n+1} are collinear if and only if $\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$ is singular. Indeed, P_0, P_n, P_{n+1} are collinear if and only if

$$0 = \begin{bmatrix} x_n & y_n \\ x_{n+1} & y_{n+1} \end{bmatrix} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}^n.$$

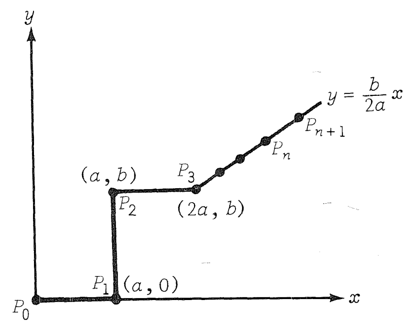

Figure 4

$ab(a - 1) \neq 0$

(iv) Suppose (Fig. 5) a nonsingular matrix U generates an area-bisecting path $P_0P_1P_2\ldots$. Then for $\theta \geq 0$, the successive rows of $\{\theta U^n : n \geq 1\}$ also produce an area-bisecting path $Q_0Q_1Q_2\ldots$, where $Q_n = \theta P_n$ for each $n \geq 1$. However, for $\theta \neq 1$, the path $Q_0Q_1Q_2\ldots$ cannot be matrix generated since $\theta U^n = (\theta U)^n$ for all $n \geq 1$ requires that U be singular.

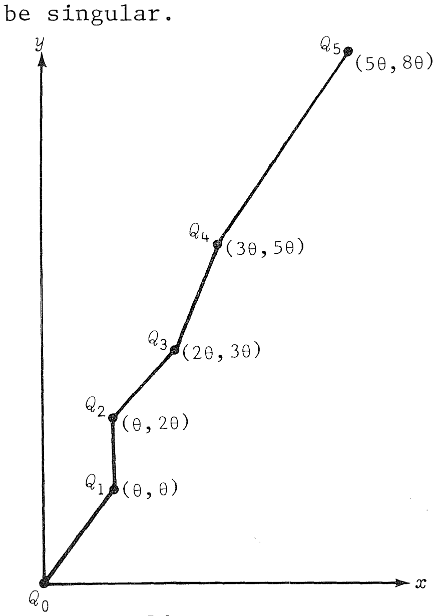

Figure 5

$Q_n = \theta F_n = (\theta f_{n-1}, \theta f_n);\ \theta(\theta - 1) \neq 0$

Under what conditions on the entries of a 2×2 real, nonnegative matrix U will the successive rows of U^n generate the consecutive vertices of an area-bisecting k-path? By definition, the path $P_0P_1P_2\ldots$ is generated by $\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$ if and only if

$$\begin{bmatrix} x_{2n-1} & y_{2n-1} \\ x_{2n} & y_{2n} \end{bmatrix} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}^n \quad \text{for each } n \geq 1.$$

This is equivalent to

$$\text{(11)} \quad \begin{bmatrix} x_{2n+1} & y_{2n+1} \\ x_{2n+2} & y_{2n+2} \end{bmatrix} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} \begin{bmatrix} x_{2n-1} & y_{2n-1} \\ x_{2n} & y_{2n} \end{bmatrix} \quad \text{for all } n \geq 1.$$

Thus, $P_0P_1P_2\ldots$ is generated by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$ if and only if for all $n \geq 1$:

(12a) $\quad x_{2n+1} = ax_{2n-1} + bx_{2n} \qquad y_{2n+1} = ay_{2n-1} + by_{2n}$

(12b) $\quad x_{2n+2} = cx_{2n-1} + dx_{2n} \qquad y_{2n+2} = cy_{2n-1} + dy_{2n}$

and (since path vertices are assumed to be distinct)

(12c) $\quad (x_{n+k}, y_{n+k}) \neq (x_n, y_n)$ for $k > 0$.

Note that (12c) requires that $(a, b) \neq (0, 1)$ and that $(c, d) \notin \{(a,b), (0,1)\}$.

Theorem 2: The path $P_0P_1P_2\ldots$ generated by a real, nonnegative matrix

$$U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is nondecreasing if and only if

(13) $\quad a \leq c \leq a^2 + bc \quad$ and $\quad b \leq d \leq ab + bd.$

A nondecreasing U-generated path is an area-bisecting k-path if and only if:

(i) $\quad |U| = 0$

or

(14) $\quad$ (ii) $\quad (1 - a)\sum_{t=1}^{m} |U|^t = 0 \quad$ for $k = 2m$.

Proof: Since

$$U^2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix},$$

the conditions in (13) are necessary for the U-generated path to be nondecreasing. To see that they are also sufficient, let $P_n = (x_n, y_n)$ for $n \geq 0$. Then (13) yields $x_2 \geq x_1 \geq 0$ and $y_2 \geq y_1 \geq 0$. From (12a) and (12b), we also obtain

$$x_{2n+2} - x_{2n+1} = (c - a)x_{2n-1} + (d - b)x_{2n} \geq 0$$

and

$$\begin{aligned} x_{2n+1} - x_{2n} &= ax_{2n-1} + (b - 1)x_{2n} \\ &= a(ax_{2n-3} + bx_{2n-2}) + (b - 1)(cx_{2n-3} + dx_{2n-2}) \\ &= (a^2 + bc - c)x_{2n-3} + (ab + bd - d)x_{2n-2} \\ &\geq 0. \end{aligned}$$

Thus, $x_{2n+2} \geq x_{2n+1} \geq x_{2n}$ for all $n \geq 0$. A similar argument establishes that $y_{2n+2} \geq y_{2n+1} \geq y_{2n}$ for all $n \geq 0$.

For the nondecreasing U-generated path, (12a) yields

$$\begin{vmatrix} x_{2n} & y_{2n} \\ x_{2n+1} & y_{2n+1} \end{vmatrix} = \begin{vmatrix} x_{2n} & y_{2n} \\ ax_{2n-1} + bx_{2n} & ay_{2n-1} + by_{2n} \end{vmatrix} = -a\begin{vmatrix} x_{2n-1} & y_{2n-1} \\ x_{2n} & y_{2n} \end{vmatrix}$$

for all $n \geq 1$. Thus,

(15) $$\begin{vmatrix} x_{2n-1} & y_{2n-1} \\ x_{2n} & y_{2n} \end{vmatrix} = |U|^n \quad \text{and} \quad \begin{vmatrix} x_{2n} & y_{2n} \\ x_{2n+1} & y_{2n+1} \end{vmatrix} = -a|U|^n$$

for all $n \geq 1$. We now use (15) to simplify (4). For $k = 2m$, condition (4) reduces to

$$(1 - a)|U|^{nm} \cdot \sum_{t=1}^{m} |U|^t = 0 \quad \text{for all } n \geq 0.$$

This is equivalent to (14). For $k = 2m + 1$, condition (4) reduces to

(16a) $$|U|^{nk/2} \cdot \left\{(1 - a)\sum_{t=1}^{m} |U|^t + |U|^{m+1}\right\} = 0 \quad (n \text{ even})$$

(16b) $$|U|^{n(k+1)/2} \cdot \left\{(1 - a)\sum_{t=1}^{m} |U|^t - a\right\} = 0 \quad (n \text{ odd}).$$

Conditions (16a) and (16b) can hold for all $n \geq 0$ only if $|U| = 0$. Indeed, $|U| \neq 0$ ensures that $a > 0$ and that $|U|^{m+1} = -a < 0$. But then, by equating the formulas for a in (16a) and (16b), we obtain the contradiction

$$a = |U|/a \quad \text{or} \quad |U| = a^2 > 0.$$

Corollary 2.1: Let $S_n = (s_n, s_{n+1})$ for the positive sequence

$$s_1,\ s_2, \text{ and } s_n = \beta s_{n-1} + \alpha s_{n-2} \quad (n \geq 3) \quad \text{[given as (5) above]}.$$

Then $S_0S_1S_2\ldots$ is a matrix-generated, area-bisecting k-path if and only if:

(17a) (i) k is even and $\beta = s_2 \geq s_1 = \alpha = 1$ (for $s_2^2 \neq s_1s_3$) in which case

(17b) $$\begin{bmatrix} s_{2n-1} & s_{2n} \\ s_{2n} & s_{2n+1} \end{bmatrix} = \begin{bmatrix} 1 & \beta \\ \beta & \beta^{n+1} \end{bmatrix} \quad (n \geq 1);$$

or

(18a) (ii) $s_1^2 + s_2^2 = s_3$ and $s_2 \neq s_1$ (for $s_2^2 = s_1s_3$), in which case

(18b) $$\begin{bmatrix} s_{2n-1} & s_{2n} \\ s_{2n} & s_{2n+1} \end{bmatrix} = \left(\frac{s_2}{s_1}\right)^{2n-2} \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} \quad (n \geq 1).$$

Proof: An inductive argument, beginning with

$$[s_2 \quad s_3] = [s_1 \quad s_2]\begin{bmatrix} 0 & \alpha \\ 1 & \beta \end{bmatrix}$$

and

$$[s_3 \quad s_4] = [s_2 \quad s_3]\begin{bmatrix} 0 & \alpha \\ 1 & \beta \end{bmatrix} = [s_1 \quad s_2]\begin{bmatrix} 0 & \alpha \\ 1 & \beta \end{bmatrix}^2,$$

establishes that

$$[s_n \quad s_{n+1}] = [s_1 \quad s_2]\begin{bmatrix} 0 & \alpha \\ 1 & \beta \end{bmatrix}^{n-1} \quad (n \geq 2).$$

In particular,

$$[s_{2n-1} \quad s_{2n}] = [s_1 \quad s_2]\begin{bmatrix} 0 & \alpha \\ 1 & \beta \end{bmatrix}^{2n-2}$$

$$[s_{2n} \quad s_{2n+1}] = [s_1 \quad s_2]\begin{bmatrix} 0 & \alpha \\ 1 & \beta \end{bmatrix}^{2n-1} = [s_2 \quad s_3]\begin{bmatrix} 0 & \alpha \\ 1 & \beta \end{bmatrix}^{2n-2}$$

can be recast in matrix form as

$$\begin{bmatrix} s_{2n-1} & s_{2n} \\ s_{2n} & s_{2n+1} \end{bmatrix} = \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}\begin{bmatrix} 0 & \alpha \\ 1 & \beta \end{bmatrix}^{2n-2} = \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}\begin{bmatrix} \alpha & \alpha\beta \\ \beta & \beta^2 + \alpha \end{bmatrix}^{n-1}$$

for all $n \geq 2$. Therefore, $S_0S_1S_2\ldots$ is $\begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}$-generated if and only if

$$\begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}^n = \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}\begin{bmatrix} \alpha & \alpha\beta \\ \beta & \beta^2 + \alpha \end{bmatrix}^{n-1} \quad \text{for all } n \geq 1. \tag{19}$$

(i) Assume that $s_2^2 \neq s_1s_3$. If $S_0S_1S_2\ldots$ is a matrix-generated, area-bisecting path, then (Corollary 1.1) k is even and $\alpha = 1$. Setting $n = 2$ in (19) and pre-multiplying by

$$\begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}^{-1},$$

we obtain

$$\begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} = \begin{bmatrix} \alpha & \alpha\beta \\ \beta & \beta^2 + \alpha \end{bmatrix}.$$

Since $\alpha = 1$, we see that $\beta = s_2$ and (17a) holds. Conversely, (17a) yields

$$\begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} = \begin{bmatrix} 1 & \beta \\ \beta & \beta^2 + 1 \end{bmatrix}$$

and therefore (19). Since (17a) ensures that (13) and (14) hold (since $a = s_1 = 1$), the path $S_0S_1S_2\ldots$ is also area-bisecting.

(ii) Assume that $s_2^2 = s_1s_3$. Then (Corollary 1.1) the straight-line path $S_0S_1S_2\ldots$ is area-bisecting. Moreover, by (11), the path $S_0S_1S_2\ldots$ is generated by

$$\begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}$$

if and only if

$$\begin{bmatrix} s_{2n+1} & s_{2n+2} \\ s_{2n+2} & s_{2n+3} \end{bmatrix} = \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}\begin{bmatrix} s_{2n-1} & s_{2n} \\ s_{2n} & s_{2n+1} \end{bmatrix}$$

for all $n \geq 1$. Since $s_2^2 = s_1s_3$ is equivalent to (7), conditions (12a), (12b) reduce to

(20a) $s_{2n+1} = s_1 s_{2n-1} + s_2 s_{2n}$,

(20b) $s_{2n+2} = s_1 s_{2n} + s_2 s_{2n+1}$,

(20c) $s_{2n+3} = s_2 s_{2n} + s_3 s_{2n+1}$,

(20d) $s_{n+1} \neq s_n$,

for all $n \geq 1$. From (20a) and (20d), we obtain the necessary condition (18a) for $S_0 S_1 S_2 \ldots$ to be matrix generated. The condition $s_1^2 + s_2^2 = s_3$ in (18a) is also sufficient since, in the presence of (7), conditions (20a)-(20c) are equivalent for each fixed $n_0 \geq 1$, and condition (20c) holds for n_0 if and only if condition (20a) holds for $n_0 + 1$. Since (18a) satisfies (20a) for $n_0 = 1$, it follows that (20a)-(20c) hold for all $n \geq n_0 = 1$. This ensures that (12a)-(12c) hold for all $n \geq 1$, which means that $S_0 S_1 S_2 \ldots$ is generated by

$$\begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}.$$

Finally, condition (18a) also ensures that

$$\begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}^2 = \left(\frac{s_2}{s_1}\right)^2 \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}$$

and (by induction) that (18b) holds.

Example 2: A nondecreasing path generated by matrix U that satisfies $(1 - a)|U| = 0$ is an area-bisecting $2N$-path for each natural number N. Since (14) holds for $|U| = -1$ when m is even, all nonnegative real matrices

$$\begin{bmatrix} a & b \\ c & (bc - 1)/a \end{bmatrix} \quad (a \neq 0) \quad \text{and} \quad \begin{bmatrix} 0 & b \\ 1/b & d \end{bmatrix} \quad (d \geq b > 1)$$

having determinantal value -1 that satisfy (12a)-(12c) and (13) also generate area-bisecting $4N$-paths for each natural number N. Thus,

$$\begin{bmatrix} 3 & 1 \\ 7 & 2 \end{bmatrix}, \quad \begin{bmatrix} 3 & 2 \\ 4 & 7/3 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 & 2 \\ 1/2 & 3 \end{bmatrix}$$

generate area-bisecting 4 -paths ($n \geq 1$) that (Theorem 2) are not area-bisecting 2-paths.

Remarks: For the $\begin{bmatrix} 0 & 2 \\ 1/2 & 3 \end{bmatrix}$-generated path $P_0 P_1 P_2 \ldots$, let $R_k = \text{Area}\{X_{k-1} P_{k-1} P_k X_k\}$ and $L_k = \text{Area}\{Y_{k-1} P_{k-1} P_k Y_k\}$ for each $k \geq 1$. Then

$$R_k = \left\{\begin{matrix} L_k, & k \text{ odd} \\ L_k - (-1)^{k/2}, & k \text{ even} \end{matrix}\right\},$$

and $P_0 P_1 P_2 \ldots$ is an area-bisecting k-path if and only if $N = 4$ $(N \geq 1)$.

δ-Splitting k-paths

The notion of area-bisecting k-paths can be extended in several ways, the two most natural extensions being those given below as Definitions A and B. A nondecreasing path is a δ-splitting (δ ≥ 0) k-path if:

Definition A: $\text{Area}\{X_1P_1P_2\ldots P_{nk+1}X_{nk+1}\} = \delta\cdot\text{Area}\{Y_1P_1P_2\ldots P_{nk+1}Y_{nk+1}\}$ for all $n \geq 1$;

Definition B: $\text{Area}\{P_0P_1P_2\ldots P_{nk+1}X_{nk+1}\} = \delta\cdot\text{Area}\{P_0P_1P_2\ldots P_{nk+1}Y_{nk+1}\}$ for all $n \geq 1$.

Although these definitions are equivalent (to the area-bisecting property) when $\delta = 1$, they yield different results for $\delta \neq 1$. Beyond generalizing our results, motivation for investigating δ-splitting k-paths also comes from the following.

Example 3: Find an expression for $f(x)$ such that $p = \{(x, f(x)): x \geq 0\}$ is an increasing path characterized by

$$\text{Area}\{\mathscr{R}_x\} = \delta\cdot\text{Area}\{\mathscr{L}_x\}$$

for each point $(x, f(x)) \in p$. (See Fig. 6.)

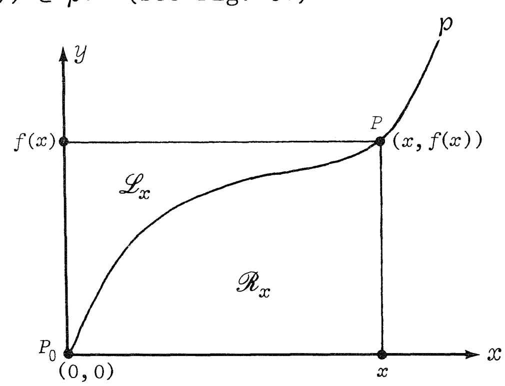

Figure 6

Our area requirement is equivalent to determining $f(x)$ such that

$$\int_0^x f(t)\,dt = \delta\cdot\int_0^{f(x)} f^{-1}(s)\,ds.$$

This can be recast as

$$(21)\qquad \left(1 + \frac{1}{\delta}\right)\int_0^x f(t)\,dt = xf(x).$$

Differentiating (21) with respect to x and rearranging terms, we obtain

$$\frac{f'(x)}{f(x)} = \frac{1}{\delta x}.$$

Thus, for arbitrary positive constant C,

$$(22)\qquad f(x) = Cx^{1/\delta}.$$

For $\delta = 1$, we obtain the area-bisecting linear paths $f(x) = Cx$. Equation (22) also provides some means for constructing approximate δ-splitting k-paths. Let $\Delta(a, b)$ denote the shaded region in Figure 7. Then the trapezoidal areas $R_a = X_aP_aP_bX_b$ and $L_a = Y_aP_aP_bY_b$ satisfy

$$(23)\qquad |\text{Area}\{R_a\} - \delta\cdot\text{Area}\{L_a\}| = (1 + \delta)\Delta(a, b).$$

To approximate a δ-splitting k-path, sum (23) over j consecutive points ($j = nk + 1$ for Def. A; $j = nk + 2$ for Def. B), and obtain

$$(24)\qquad \left|\text{Area}\left\{\sum_{i=1}^{j} R_{a_i}\right\} - \delta\cdot\text{Area}\left\{\sum_{i=1}^{j} L_{a_i}\right\}\right| \leq (1 + \delta)\sum_{i=1}^{j}\Delta(a_i, b_i).$$

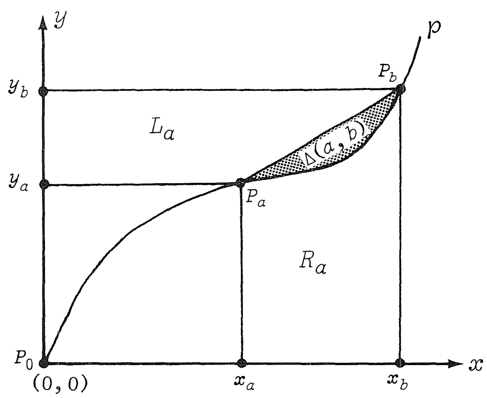

Figure 7

By way of illustration, consider $f(x) = x^2$ for the case where $\delta = 1/2$. A straightforward computation yields $\Delta(a, b) = (b - a)^3/6$ for consecutive points $P_i = (x_i, x_i^2)$ with $x_{i+1} - x_i = b - a$. Since Area$\{R_{x_i}\}$ = Area$\{L_{x_i}\}$ + $(b - a)^3/6$ for each such sector,

$$\text{Area}\left\{\sum_{i=1}^{j} R_{x_i}\right\} = \frac{1}{2}\cdot \text{Area}\left\{\sum_{i=1}^{j} L_{x_i}\right\} + \frac{j(b - a)^3}{12}.$$

For j such consecutive points, the error $E(j, a, b) = j(b - a)^3/12$ in constructing a 1/2-splitting k-path can be made less than ε by choosing the points on $y = x^2$ such that $x_{i+1} - x_i < (12\varepsilon/j)^{1/3}$ for each $i = 1, 2, \ldots, j - 1$.

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