SOME PROPERTIES OF THE TETRANACCI SEQUENCE MODULO m

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First Wall [8] and subsequently a number of others (see, e.g., [1], [3], and [4]) have examined the properties of the Fibonacci sequence modulo m. The Tribonacci sequence modulo m was considered and a number of properties were derived in [6]. Chang [2] briefly examined higher-order sequences modulo m. Vince [5] considered the period of repetition of a general linear recurrence.

In this paper we list several basic results which follow when some of Vince's results are applied to the special case of the Tetranacci sequence. We then establish a number of additional properties. We also investigate in detail the relationship of the period of the Tetranacci sequence modulo m to the factorization of the minimum polynomial of the T-matrix defined in [7] and given below in (2).

We consider the sequence $\{M_n\}$ reduced modulo m, taking least nonnegative residues, where

(1) $\quad M_n = M_{n-1} + M_{n-2} + M_{n-3} + M_{n-4}\ (n \geq 4),\ M_0 = M_1 = 0,\ M_2 = M_3 = 1.$

Definitions: The *length of the period* of $\{M_n\}$ (mod m), designated $h(m)$, is the number of terms in one period of the sequence $\{M_n\}$ (mod m). A *simply periodic* sequence is periodic and repeats by returning to its initial values.

We list several results found in [7] which will be required in the development of this paper.

(2) $\quad$ (a) $$T^n = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}^n = \begin{bmatrix} M_{n+2} & N_{n+2} & S_{n+2} & M_{n+1} \\ M_{n+1} & N_{n+1} & S_{n+1} & M_n \\ M_n & N_n & S_n & M_{n-1} \\ M_{n-1} & N_{n-1} & S_{n-1} & M_{n-2} \end{bmatrix},$$

where

(3) $\quad N_n = M_{n-1} + M_{n-2} + M_{n-3},$

(4) $\quad S_n = M_{n-1} + M_{n-2}.$

(5) $\quad$ (b) $|T| = -1$, where $|T|$ is the determinant of T.

(6) $\quad$ (c) $$|T^n| = \begin{vmatrix} M_{n+3} & M_{n+2} & M_{n+1} & M_n \\ M_{n+2} & M_{n+1} & M_n & M_{n-1} \\ M_{n+1} & M_n & M_{n-1} & M_{n-2} \\ M_n & M_{n-1} & M_{n-2} & M_{n-3} \end{vmatrix} = (-1)^n.$$

(7) $\quad$ (d) $M_{n+p} = M_{n+i}M_{p-i+2} + M_{n+i-1}N_{p-i+2} + M_{n+i-2}S_{p-i+2} + M_{n+i-3}M_{p-i+1}.$

(8) $\quad$ (e) $$\sum_{i=0}^{n} M_i = \frac{1}{3}(M_{n+2} + 2M_n + M_{n-1} - 1).$$

(9) $\quad$ (f) $$\sum_{i=0}^{n} M_{2i+1} = \frac{1}{3}(2M_{2n+2} + M_{2n} - M_{2n-1} - 2).$$

Table 1 gives values of $h(m)$ for selected values of m.

Table 1

m	2	3	4	5	6	7	8	9	10	11	13	15	16	17	19	27	100
$h(m)$	5	26	10	312	130	342	20	78	1560	120	84	312	40	4912	6858	234	1560

Results in [5] may be applied to the Tetranacci sequence to yield the following theorem.

Theorem 1: The sequence $\{M_n\}$ (mod m) satisfies the following:

(a) The sequence $\{M_n\}$ (mod m) is simply periodic.

(b) If m has prime factorization $m = p_1^{t_1}p_2^{t_2} \ldots p_s^{t_s}$, then
$$h(m) = \mathrm{LCM}[h(p_1^{t_1}), h(p_2^{t_2}), \ldots, h(p_s^{t_s})].$$

(c) If $h(p^2) \neq h(p)$, then $h(p^t) = p^{t-1}h(p)$.

(d) If $n > 0$ is least such that $M_{n+1} \equiv M_n \equiv M_{n-1} \equiv 0 \pmod m$, and if $M_{t+1} \equiv M_t \equiv M_{t-1} \equiv 0 \pmod m$, then $t = kn$ for some integer k.

If we examine the terms of $\{M_n\}$ (mod 5), we see that for $s = 78$ we have
$$M_{s-1} \equiv M_s \equiv M_{s+1} \equiv 0 \pmod 5,$$
but $M_{s-2} \not\equiv 1 \pmod 5$. Hence, s is not the length of the period of $\{M_n\}$ (mod 5). However, the occurrence of "triple zeros," 0, 0, 0, in $\{M_n\}$ (mod 5) and, in general, the occurrence of triple zeros in the sequence (mod m), is significant in determining, among other properties, the period structure. The following lemma states some of the results related to this phenomenon.

Lemma 1: If $s > 0$ is least such that
$$M_{s-1} \equiv M_s \equiv M_{s+1} \equiv 0 \pmod m,$$
then the following congruences are valid:

(a) $M_{s-2}^8 \equiv M_{s+2}^8 \equiv 1 \pmod m$,

(b) $M_{js-1} \equiv M_{js} \equiv M_{js+1} \equiv 0 \pmod m$ for all $j > 0$.

Proof: To prove (a), we use (6) to obtain
$$(-1)^s = |T^s| = \begin{vmatrix} M_{s+3} & M_{s+2} & M_{s+1} & M_s \\ M_{s+2} & M_{s+1} & M_s & M_{s-1} \\ M_{s+1} & M_s & M_{s-1} & M_{s-2} \\ M_s & M_{s-1} & M_{s-2} & M_{s-3} \end{vmatrix} \equiv \begin{vmatrix} M_{s+3} & M_{s+2} & 0 & 0 \\ M_{s+2} & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{s-2} \\ 0 & 0 & M_{s-2} & M_{s-3} \end{vmatrix}$$
$$\equiv M_{s+2}^2 M_{s-2}^2 \equiv M_{s+2}^4 \pmod m.$$
Therefore, $M_{s+2}^4 \equiv \pm 1 \pmod m$ or $M_{s+1}^8 \equiv 1 \pmod m$ and the proof is complete.

In (b), we prove only that $M_{js} \equiv 0 \pmod m$. The other parts follow similarly. The proof is by induction on j. If $j = 1$, the result is clear. If we assume that $M_{js} \equiv 0 \pmod m$, we have, by (7) with $i = 1$,
$$M_{(j+1)s} = M_{js+s} = M_{js+1}M_{s+1} + M_{js}N_{s+1} + M_{js-1}S_{s+1} + M_{js-2}M_s \equiv 0 \pmod m,$$
and the induction is complete.

The next theorem provides identities which involve a rather curious shift of a factor of the *subscript* of an appropriate M_n to a *power* of that M_n when the modulus is changed from m to m^2.

Theorem 2: Let $h = h(m)$ and let k be a positive integer, then the following identities hold.

(10) (a) $M_{kh-2} \equiv M_{h-2}^k \pmod{m^2}$,

(11) (b) $M_{kh-1} \equiv kM_{h-2}^{k-1}M_{h-1} \pmod{m^2}$,

(12) (c) $M_{kh} \equiv kM_{h-2}^{k-1}M_h \pmod{m^2}$,

(13) (d) $M_{kh+1} \equiv kM_{h-2}^{k-1}M_{h+1} \pmod{m^2}$.

We prove (10); the other parts follow similarly. The proof is by induction on k. If $k = 1$, the conclusion is immediate. If we assume that

$$M_{kh-2} \equiv M_{h-2}^k \pmod{m^2},$$

then, by (7) with $i = 2$ and the induction hypothesis,

$$M_{(k+1)h-2} = M_{(kh-2)+h} = M_{kh}M_h + M_{kh-1}N_h + M_{kh-2}S_h + M_{kh-3}M_{h-1}$$

$$(\mathrm{mod}\ m^2)\begin{cases} \equiv [M_{kh-1}(M_{h-2} + M_{h-3}) + M_{h-2}^kM_{h-2} + M_{h-1}(M_{kh-2} + M_{kh-3})] \\ \equiv [M_{kh-1}(M_{h+1} - M_h - M_{h-1}) + M_{h-2}^{k+1} + M_{h-1}(M_{kh+1} - M_{kh} - M_{kh-1})] \\ \equiv M_{h-2}^{k+1}, \end{cases}$$

since m divides M_{h+1}, M_h, M_{h-1} and, by Lemma 1(b), m divides M_{kh+1}, M_{kh}, M_{kh-1}. This completes the proof.

A related property is the following:

Lemma 2: If p is prime and $j = h(p^t)$ is the length of the period of $\{M_n\}$ (mod p^t), then

$$M_{j-2}^p \equiv 1 \pmod{p^{t+1}}.$$

Proof: Since $M_{j-2} \equiv 1 \pmod{p^t}$, $M_{j-2} \equiv 1 \pmod{p}$, and thus $M_{j-2}^s \equiv 1 \pmod{p}$ for all s. Consequently, we have

$$\begin{aligned}(M_{j-2}^p - 1) &= (M_{j-2} - 1)(M_{j-2}^{p-1} + M_{j-2}^{p-2} + \cdots + M_{j-2} + 1) \\ &\equiv [0 \pmod{p^t}][(1 + 1 + \cdots + 1) \pmod{p}] \\ &\equiv [0 \pmod{p^t}][0 \pmod{p}] \\ &\equiv 0 \pmod{p^{t+1}}.\end{aligned}$$

The occurrence in $\{M_n\}$ (mod m) of the quadruple 1, 0, 0, 0 is the signal that the end of the period has been reached and that repetition has begun. If the term immediately in front of the three zeros is M_{s-2}, where $M_{s-2} \not\equiv 1$ (mod m), there are only a limited number of possibilities for the value of M_{s-2} since, by Lemma 1, we always have $M_{s-2}^8 \equiv 1 \pmod{m}$. This implies that as an element of the finite group, $\mathbb{Z}_m$, the order of M_{s-2} is 2, 4, or 8. We now examine in detail the possibilities resulting from this implication.

Theorem 3: If s is least such that

$$M_{s-1} \equiv M_s \equiv M_{s+1} \equiv 0 \pmod{m} \quad \text{and} \quad M_{s-2} \not\equiv 1 \pmod{m},$$

then one of the following holds:

(a) If the order of $M_{s-2} = 2$, then $M_{2s-2} \equiv M_{s-2}^2 \equiv 1 \pmod{m}$ and $h(m) = 2s$. An example is $\{M_n\}$ (mod 31), where $s = 30{,}784$ and $h(31) = 61{,}568$.

(b) If the order of $M_{s-2} = 4$, then $M_{4s-2} \equiv M_{s-2}^4 \equiv 1 \pmod{m}$ and $h(m) = 4s$. An example is $\{M_n\}$ (mod 5), where $s = 78$ and $h(5) = 312$.

(c) If the order of $M_{s-2} = 8$, then $M_{8s-2} \equiv M_{s-2}^8 \equiv 1 \pmod{m}$ and $h(m) = 8s$. An example is $\{M_n\}$ (mod 89), where $s = 1165$ and $h(89) = 9320$.

Proof: The proof follows from Theorem 2 and from the fact that, if $a \equiv b \pmod{m^2}$, then $a \equiv b \pmod{m}$.

The following theorem gives further related results.

Theorem 4: If s is least such that

$$M_{s-1} \equiv M_s \equiv M_{s+1} \equiv 0 \pmod{m},$$

then one of the following holds:

(a) If $h(m) = 2s$, then for any r, $M_r + M_{r+s} \equiv 0 \pmod{m}$.

(b) If $h(m) = 4s$, then for any r, $M_r + M_{r+s} + M_{r+2s} + M_{r+3s} \equiv 0 \pmod{m}$.

(c) If $h(m) = 8s$, then for any r, $M_r + M_{r+s} + M_{r+2s} + \cdots + M_{r+7s} \equiv 0 \pmod{m}$.

Proof: We prove (b); the other parts follow similarly. By repeated use of (7) with $i = 1$, we have

$$\begin{aligned}
&M_r + M_{r+s} + M_{r+2s} + M_{r+3s} \\
&= M_r + (M_{r+1}M_{s+1} + M_rN_{s+1} + M_{r-1}S_{s+1} + M_{r-2}M_s) \\
&\quad + (M_{r+1}M_{2s+1} + M_rN_{2s+1} + M_{r-1}S_{2s+1} + M_{r-2}M_{2s}) \\
&\quad + (M_{r+1}M_{3s+1} + M_rN_{3s+1} + M_{r-1}S_{3s+1} + M_{r-2}M_{3s}) \\
&\equiv M_r(1 + M_{s-2} + M_{2s-2} + M_{3s-2}) \pmod{m} \\
&\equiv M_r(1 + M_{s-2} + M_{s-2}^2 + M_{s-2}^3) \pmod{m} \\
&\equiv 0 \pmod{m}
\end{aligned}$$

since $M_{s-2}^4 - 1 \equiv 0 \pmod{m}$ and $M_{s-2} - 1 \not\equiv 0 \pmod{m}$.

Remark: The preceding proof shows that under the hypotheses of (b),

$$\begin{aligned}
M_{r+s} &\equiv M_rM_{s-2} \pmod{m}, \\
M_{r+2s} &\equiv M_rM_{2s-2} \equiv M_rM_{s-2}^2 \pmod{m}, \\
M_{r+3s} &\equiv M_rM_{3s-2} \equiv M_rM_{s-2}^3 \pmod{m},
\end{aligned}$$

whenever $M_{s+1} \equiv M_s \equiv M_{s-1} \equiv 0 \pmod{m}$.

From these congruences we conclude that whenever triple zeros, 0, 0, 0, appear in the interior of the period rather than at the end, the triple zeros divide the period into what we might call *subperiods* of equal length where the terms in each successive subperiod are a fixed multiple of the corresponding terms in the first subperiod; that is, the terms which precede the first set of triple zeros.

For example, in the sequence $\{M_n\}$ (mod 5), we have 0, 0, 0 as the terms with subscripts 77, 78, 79; 155, 156, 157; 233, 234, 235; 311, 312, 313. If we call the first 78 (length of the subperiod) terms A, then the second 78 terms are obtained as 3 times A, the third 78 as $3^2 \equiv 4 \pmod{5}$ times A, and the fourth as $3^3 \equiv 2 \pmod{5}$ times A. Further, we have $3^4 \equiv 1 \pmod{5}$ and the length of the period is $4 \times 78 = 312$.

Theorem 5: For $p > 2$, $h(p)$ is even.

Proof: Let $h = h(p)$ and use (6) to obtain

$$(-1)^h = \begin{vmatrix} M_{h+3} & M_{h+2} & M_{h+1} & M_h \\ M_{h+2} & M_{h+1} & M_h & M_{h-1} \\ M_{h+1} & M_h & M_{h-1} & M_{h-2} \\ M_h & M_{h-1} & M_{h-2} & M_{h-3} \end{vmatrix} \equiv \begin{vmatrix} M_{h+3} & M_{h+2} & 0 & 0 \\ M_{h+2} & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{h-2} \\ 0 & 0 & M_{h-2} & M_{h-3} \end{vmatrix}$$

$$\equiv M_{h-2}^2 M_{h+2}^2 \equiv M_{h+2}^4 \equiv M_{h+2} \equiv 1 \pmod{p}.$$

Therefore, $(-1)^h \equiv 1 \pmod{p}$ and h is even.

Theorem 6: If $p > 2$ and s is least such that

$$M_{s-1} \equiv M_s \equiv M_{s+1} \equiv 0 \pmod{p},$$

but $M_{s-2} \not\equiv 1 \pmod{p}$, then one of the following holds:

(a) If $h(p) = 2s$, then s is even.
(b) If $h(p) = 4s$, then s is even.
(c) If $h(p) = 8s$, then s is odd.

Proof: We prove (c); the other parts follow similarly. If $h(p) = 8s$, then by Theorem 3(c), $M_{s+2}^8 \equiv 1 \pmod{p}$, which implies that $M_{s+2}^4 \equiv (-1) \pmod{p}$. But, from the proof of Theorem 4, $M_{s+2}^4 \equiv (-1)^s \pmod{p}$ also. Hence, $(-1) \equiv (-1)^s \pmod{p}$ and s is odd.

We now examine further the relationship of p to $h(p)$. The minimum polynomial of the matrix T,

$$f(x) = x^4 - x^3 - x^2 - x - 1,$$

and its factorization over Z_p determine what this relationship is. We begin by stating a theorem that follows from more general results in [5].

Theorem 7: If

$$f(x) = x^4 - x^3 - x^2 - x - 1 = g_1^{\alpha_1}(x)g_2^{\alpha_2}(x)g_3^{\alpha_3}(x)g_4^{\alpha_4}(x)$$

is the factorization into irreducible factors of $f(x)$ over Z_p, then

(a) $h(p)\,|\,p^s\mathrm{LCM}[t_i(p^{m_i} - 1)/(p - 1)]$, where s satisfies $p^s \geq \max \alpha_i > p^{s-1}$, m_i is the degree of $g_i(x)$, and t_i is the multiplicative order of $b_i(-1)^{m_i}$ in Z_p, b_i being the constant term of $g_i(x)$.

(b) If $t_i^r\,|\,(p^{m_i} - 1)/(p - 1)$ for some integer r, then $t_i^{r+1}\,|\,h(p)$.

We now apply Theorem 7 to the cases that arise from possible factorizations of $f(x)$.

Case 1. $f(x)$ is irreducible. In this case we have $m_1 = 4$, $\alpha_2 = 1$, $s = 0$, $t_1 = 2$, $r = 2$.

Hence, $h(p)\,|\,2(p^3 + p^2 + p + 1)$ and $8\,|\,h(p)$. An example is $p = 5$, where $h(5) = 312$, which divides $2(5^3 + 5^2 + 5 + 1) = 312$ and is divisible by 8.

Case 2. $f(x)$ has a single linear factor.

We then have $m_1 = 1$, $m_2 = 3$, $\alpha_1 = \alpha_2 = 1$, $s = 0$; t_1, $t_2\,|\,p - 1$,

$$h(p)\,|\,\mathrm{LCM}[t_1,\ t_2(p^2 + p + 1)]$$

and

$$t_1\,|\,h(p) \quad \text{and if } t_2^r\,|\,p^2 + p + 1, \quad \text{then } t_2^{r+1}\,|\,h(p).$$

An example is $p = 3$, where

$$f(x) = (x - 1)(x^3 - x + 1)$$

and $h(3) = 26$, $t_1 = 1$, $t_2 = 2$, $r = 0$. Then $26\,|\,2(3^2 + 3 + 1)$ and $2\,|\,26$.

Case 3. $f(x)$ has exactly two distinct linear factors.

We then have $m_1 = m_2 = 1$, $m_3 = 2$, $\alpha_1 = \alpha_2 = \alpha_3 = 1$, $s = 0$; t_1, t_2, $t_3\,|\,p - 1$, $h(p)\,|\,p^2 - 1$ and if $t_3^r\,|\,p + 1$ for some integer r, then $t_3^{r+1}\,|\,h(p)$.

An example is $p = 29$, where

$$f(x) = (x - 7)(x - 15)(x^2 - 8x + 2),$$

and $h(29) = 280$, $t_1 = 7$, $t_2 = 28$, $t_3 = 4$, LCM$[7, 28, 4 \cdot 30] = 840$, which is divisible by 280. Also, 7, 28, and 4 all divide 280 and are the highest such powers.

Case 4. $f(x)$ has exactly four distinct linear factors.

We then have $m_1 = m_2 = m_3 = m_4 = 1$; $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$, $s = 0$,

$$h(p) \mid \text{LCM}[t_1, t_2, t_3, t_4]$$

and

$$t_i \mid p - 1 \text{ for } i = 1, 2, 3, 4.$$

An example is $p = 137$, where $h(137) = 136$ and

$$f(x) = (x - 40)(x - 52)(x - 58)(x - 125).$$

All the $t_i = 136$, so $h(137) \mid 136$ and all $t_i \mid 136$ as well.

Case 5. $f(x)$ has a repeated linear factor and two other distinct linear factors.

Then $m_1 = m_2 = m_3 = 1$, $\alpha_1 = \alpha_2 = 1$, $\alpha_3 = 2$, $s = 1$,

$$h(p) \mid \text{LCM}[t_1, t_2, t_3]$$

and

$$t_i \mid h(p).$$

In looking for an example of this case, we consider the discriminant of $f(x) = -563$, a prime. Therefore, this case can occur only for $p = 563$. It does, in fact, occur when $p = 563$, $h(563) = 316{,}406$, and we have

$$f(x) = (x - 107)(x - 116)(x + 111)^2.$$

Then $t_1 = t_2 = t_3 = 562$ and $h(563) \mid 563 \cdot 562 = 316{,}406$. This is the only case, of course, where $f(x)$ has a repeated root.

Case 6. $f(x)$ has two distinct quadratic factors.

Then $m_1 = m_2 = 2$; $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$, $s = 0$,

$$h(p) \mid \text{LCM}[t_1(p + 1), t_2(p + 1)]$$

and

$$t_i^{r+1} \mid h(p) \quad \text{if } t_i^r \mid p + 1.$$

Our example in this case is $p = 13$, where $h(13) = 84$ and

$$f(x) = (x^2 + 4x - 3)(x^2 - 5x - 4).$$

Then $t_1 = t_2 = 6$, with $r = 0$, $h(13) \mid 84$, and $6 \mid 84$.

These six are the only possible cases because all other factorizations of $f(x)$ can be shown to be untenable.

Table 2 gives additional examples.

Table 2

	$h(p)$	Roots of $f(x)$ in Z_p	Factorization of $f(x)$ over Z_p
7	$342 = 7^3 - 1$	5	$(x - 5)(x^3 + 4x^2 + 5x + 3)$
11	$120 = (11^4 - 1)/122$	none	irreducible
17	$4{,}912 = 17^3 - 1$	6	$(x - 6)(x^3 + 5x^2 + 12x + 3)$
41	$240 = (41^2 - 1)/7$	3, 33	$(x - 3)(x - 33)(x^2 - 6x + 12)$
43	$162{,}800 = (43^4 - 1)/21$	none	irreducible
47	$103{,}822 = 47^3 - 1$	21	$(x - 21)(x^3 + 20x^2 - 4x + 9)$
67	$100{,}254 = (67^3 - 1)/2$	5	$(x - 5)(x^3 + 4x^2 + 19x + 27)$
73	$2{,}664 = (73^3 - 1)/2$	39, 66	$(x - 39)(x - 66)(x^2 + 31x - 23)$
109	$2{,}614{,}040 = (109^4 - 1)/54$	none	irreducible

Finally, we state a theorem which gives a number of congruences involving sums.

Theorem 8: If $h = h(m)$, then the following congruences hold:

(a) $\sum_{i=0}^{h} M_i \equiv 0 \pmod{m}$, (e) $\sum_{i=0}^{h} M_{3i+1} \equiv 0 \pmod{m}$,

(b) $\sum_{i=0}^{h} M_{2i+1} \equiv 0 \pmod{m}$, (f) $\sum_{i=0}^{h-1} M_{3i+2} \equiv 0 \pmod{m}$,

(c) $\sum_{i=0}^{h} M_{2i} \equiv 0 \pmod{m}$, (g) $\sum_{i=0}^{(h-2)/2} M_{2i} \equiv 0 \pmod{m}$,

(d) $\sum_{i=0}^{h} M_{3i} \equiv 0 \pmod{m}$, (h) $\sum_{i=0}^{(h-2)/2} M_{2i+1} \equiv 0 \pmod{m}$.

Proof: The proofs follow easily from appropriate formulas which are derived in [7], two of which have been listed earlier. By (8) we have

$$\sum_{i=0}^{h} M_i = \frac{1}{3}(M_{h+2} + 2M_h + M_{h-1} - 1) \equiv 0 \pmod{m},$$

and by (9) and Lemma 1(b), we have

$$\sum_{i=0}^{h} M_{2i+1} = \frac{1}{3}(2M_{2h+2} + M_{2h} - M_{2h-1} - 2) \equiv 0 \pmod{m}.$$

The other congruences may be proved similarly.

A number of additional congruences involving sums of terms of $\{M_n\}$ may be derived, but no attempt is made at providing an exhaustive list.

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References

1. Agnes Andreassian. "Fibonacci Sequences Modulo M." *Fibonacci Quarterly 12.1* (1974):51-64.
2. Derek K. Chang. "Higher-Ordered Fibonacci Sequences Modulo m." *Fibonacci Quarterly 24.2* (1986):138-39.
3. John H. Halton. "On the Divisibility Properties of the Fibonacci Numbers." *Fibonacci Quarterly 4.3* (1966):217-39.
4. Andrew Vince. "The Fibonacci Sequence Modulo N." *Fibonacci Quarterly 16.5* (1978):403-07.
5. Andrew Vince. "Period of a Linear Recurrence." *Acta Arith. 39.4* (1981):303-11.
6. Marcellus E. Waddill. "Some Properties of a Generalized Fibonacci Sequence Modulo m." *Fibonacci Quarterly 16.4* (1978):344-53.
7. Marcellus E. Waddill. "The Tetranacci Sequence and Generalizations." *Fibonacci Quarterly 30.1* (1992):9-20.
8. D. D. Wall. "Fibonacci Series Modulo m." *Amer. Math. Monthly 67.6* (1960): 525-37.

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