ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-471 *Proposed by Andrew Cusumano & Marty Samberg, Great Neck, NY*

Starting with a sequence of four ones, build a sequence of finite differences where the number of finite differences taken at each step is the term of the sequence. That is,

S_1

1 1 1 1
1 2 3 4 5

S_2

1 1 1 1
1 2 3 4 5
1 2 4 7 11 16

S_3

1 1 1 1
1 2 3 4 5
1 2 4 7 11 16
1 2 4 8 15 26 42

Now, reverse the procedure but start with the powers of the last row of differences and continue until differences are constant. For example, if the power is two, we have

1 4 9 16 25
3 5 7 9
2 2 2

1 4 16 49 121 256 etc.
3 12 33 72 135
9 21 39 63
12 18 24
6 6

The sequence of constants obtained when the power is two is

2, 6, 20, 70, ...,

while the sequence of constants when the power is three is

6, 90, 1680, 34650, ...

Let N be the number of the term in the original difference sequence and M be the power used in forming the reversed sequence. Show that the constant term is

$$X(N, M) = \frac{(N \cdot M)!}{(N!)^M}, \quad N = 1, 2, 3, \ldots, \quad M = 2, 3, 4, \ldots .$$

For example,

$$x(2, 3) = \frac{6!}{2^3} = 90.$$

H-472 *Proposed by Paul S. Bruckman, Edmonds, WA*

Let $Z(n)$ denote the Fibonacci entry-point of the natural number n, that is, the smallest positive index t such that $n \mid F_t$. Prove that $n = Z(n)$ if and only if $n = 5$ or $n = 12 \cdot 5$, for some $u \geq 0$.

H-473 *Proposed by A. G. Schaake & J. C. Turner, Hamilton, New Zealand*

Show that the following [1, p. 98] is equivalent to Fermat's Last Theorem.

"For $n > 2$ there does not exist a positive integer triple (a,b,c) such that the two rational numbers r/s, p/q, with

$$r = c - a, \qquad p = b - 1,$$
$$s = \sum_{i=1}^{n} b^{n-i}, \qquad q = \sum_{i=1}^{n} a^{i-1}c^{n-i},$$

are penultimate and final convergents, respectively, of the simple continued fraction (having an odd number of terms) for p/q."

Reference

1. A. G. Schaake & J. C. Turner. *New Methods for Solving Quadratic Diophantine Equations (Part I and Part II)*. Research Report No. 192, Department of Mathematics and Statistics, University of Waikato, New Zealand, 1989.

Editorial comment: Please note that in the May 1992 issue of this quarterly, the first solution (A Triggy Problem), which is actually Problem 446, was erroneously identified as Problem 466.

SOLUTIONS

Sum Problem

H-435 *Proposed by Ratko Tošič, University of Novi Sad, Yugoslavia (Vol. 27, no. 5, November 1989)*

(a) Prove that, for $n \geq 1$,

$$F_{n+1} + \sum_{\substack{0 < i_1 < \cdots < i_k \leq n \\ 1 \leq k \leq n}} F_{n+1-i_k} F_{i_k - i_{k-1}} \cdots F_{i_2 - i_1} F_{i_1}$$
$$= \sum_{k=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \binom{n+1}{2k+1} \cdot 2^k,$$

where $\lfloor x \rfloor$ is the greatest integer $\leq x$.

(b) Prove that, for $n \geq 3$,

$$\sum_{\substack{0 < i_1 < \cdots < i_k \leq n \\ 1 \leq k \leq n}} (-1)^{n-k} F_{n-1-i_k} F_{i_k - i_{k-1}} \cdots F_{i_2 - i_1} F_{i_1 - 2} \cdot 2^k$$
$$= F_{n+3} + (-1)^{n+1} F_{n-3}.$$

Solution by Y. H. Harris Kwong, SUNY College at Fredonia, Fredonia, NY

(a) Let S_n denote the sum on the left of the given identity. Note that S_n can be rewritten as $\sum_{k=0}^{n} S_{n,k}$, where

$$S_{n,k} = \sum_{\substack{j_1, \ldots, j_{k+1} > 0 \\ j_1 + \cdots + j_{k+1} = n+1}} F_{j_1} F_{j_2} \cdots F_{j_{k+1}},$$

which is precisely the coefficient of x^{n+1} in

$$\left(\sum_{i=1}^{\infty} F_i x_i\right)^{k+1} = \left(\frac{x}{1 - x - x^2}\right)^{k+1}.$$

Therefore, S_n is the coefficient of x^{n+1} in

$$\sum_{k=0}^{\infty} \left(\frac{x}{1 - x - x^2}\right)^{k+1} = \frac{x}{1 - 2x - x^2} = \frac{1}{2\sqrt{2}}\left(\frac{1}{1 + (1 - \sqrt{2})x} - \frac{1}{1 - (1 - \sqrt{2})x}\right)$$

$$= \frac{1}{2\sqrt{2}} \sum_{n=0}^{\infty} [(1 + \sqrt{2})^n - (1 - \sqrt{2})^n]x^n.$$

Hence, we conclude that

$$S_n = \frac{1}{2\sqrt{2}} \sum_{j=0}^{n+1} \binom{n+1}{j} [\sqrt{2}^j - (-\sqrt{2})^j] = \sum_{k=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \binom{n+1}{2k+1} 2^k.$$

(b) Let T_n denote the sum on the left of the given identity, then

$$T_n = 2(-1)^{n+1} \sum_{k=1}^{n} T_{n,k},$$

where

$$T_{n,k} = \sum_{\substack{j_1, j_{k+1} \geq -1, j_2, \ldots, j_k > 0 \\ j_1 + \cdots + j_{k+1} = n-3}} F_{j_1} (-2F_{j_2}) \cdots (-2F_{j_k}) F_{j_{k+1}},$$

which is exactly the coefficient of x^{n-3} in

$$\left(\sum_{j=-1}^{\infty} F_j x^j\right)^2 \left(\sum_{i=1}^{\infty} -2F_i x^i\right)^{k-1} = \left(\frac{1}{x} + \frac{x}{1 - x - x^2}\right)^2 \left(\frac{-2x}{1 - x - x^2}\right)^{k-1}$$

Hence, we have

$$\sum_{n=0}^{\infty} (-1)^{n+1} T_n x^n = 2x^3 \left(\frac{1}{x} + \frac{x}{1 - x - x^2}\right)^2 \sum_{k=1}^{\infty} \left(\frac{-2x}{1 - x - x^2}\right)^{k-1}$$

$$= \frac{2x(1 - x)^2}{(1 - x - x^2)(1 + x - x^2)} = \frac{2 - 3x}{1 - x - x^2} - \frac{2 - x}{1 + x - x^2}.$$

It is clear that

$$\frac{1}{1 + x - x^2} = \frac{1}{(1 + \alpha x)(1 + \beta x)} = \frac{1}{\alpha - \beta}\left[\frac{\alpha}{1 + \alpha x} - \frac{\beta}{1 + \beta x}\right],$$

where $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$. Thus,

$$\frac{1}{1 + x - x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n [\alpha^{n+1} - \beta^{n+1}]}{\alpha - \beta} x^n = \sum_{n=0}^{\infty} (-1)^n F_{n+1} x^n,$$

which implies that

$$\frac{2 - 3x}{1 - x - x^2} - \frac{2 - x}{1 + x - x^2} = \sum_{n=0}^{\infty} [(2F_{n+1} - 3F_n) + (-1)^{n+1}(2F_{n+1} + F_n)]x^n.$$

Therefore, we conclude that for $n \geq 0$,

$$T_n = (2F_{n+1} + F_n) + (-1)^{n+1}(2F_{n+1} - 3F_n) = F_{n+3} + (-1)^{n+1} F_{n-3}.$$

Also solved by N. A. Volodin.

Mix and Match

H-454 *Proposed by Larry Taylor, Rego Park, NY*
(Vol. 29, no. 2, May 1991)

Construct six distinct Fibonacci-Lucas identities such that

(a) Each identity consists of three terms;
(b) Each term is the product of two Fibonacci numbers;
(c) Each subscript is either a Fibonacci or a Lucas number.

Solutions by Stanley Rabinowitz, Westford, MA

Solution Set 1

Hêre are six identities that meet the requested conditions, although they are probably not what the proposer intended:

$$F_{F_2}F_{F_n} + F_{F_3}F_{F_n} = F_{F_4}F_{F_n}$$
$$F_{F_2}F_{L_n} + F_{F_3}F_{L_n} = F_{F_4}F_{L_n}$$
$$F_{F_3}F_{F_n} + F_{F_4}F_{F_n} = F_{L_3}F_{F_n}$$
$$F_{F_3}F_{L_n} + F_{F_4}F_{L_n} = F_{L_3}F_{L_n}$$
$$F_{F_4}F_{F_n} + F_{L_3}F_{F_n} = F_{F_5}F_{F_n}$$
$$F_{F_4}F_{L_n} + F_{L_3}F_{L_n} = F_{F_5}F_{L_n}$$

Solution Set 2

If numerical identities are acceptable, then we have the following identities (found by computer search):

$$F_2F_3 + F_4F_8 = F_5F_7$$
$$F_2F_8 + F_5F_{11} = F_3F_{13}$$
$$F_2F_{18} + F_5F_{11} = F_7F_{13}$$
$$F_3F_7 + F_4F_8 = F_2F_{11}$$
$$F_3F_{13} + F_8F_{18} = F_5F_{21}$$
$$F_5F_{21} + F_8F_{34} = F_{13}F_{29}$$
$$F_8F_{18} + F_{11}F_{21} = F_3F_{29}$$
$$F_{13}F_{29} + F_{18}F_{34} = F_5F_{47}$$

where all the subscripts are distinct in each example.

Solution Set 3

The numerical identities in Solution Set 2 suggest the following identities involving one parameter, i:

$$\begin{cases} F_{F_{i+4}}F_{L_{i+1}} + F_{F_{i+2}}F_{L_{i+2}} = F_{F_i}F_{L_{i+3}} & \text{if } i \text{ is not divisible by } 3 \\ F_{F_{i+4}}F_{L_{i+1}} = F_{F_{i+2}}F_{L_{i+2}} + F_{F_i}F_{L_{i+3}} & \text{if } 3\mid i. \end{cases}$$

We will prove these by proving the equivalent single condition:

(1) $$F_{F_{i+4}}F_{L_{i+1}} - (-1)^{F_i}F_{F_{i+2}}F_{L_{i+2}} = F_{F_i}F_{L_{i+3}}.$$

To verify identity (1), we apply the known transformation

$$5F_m F_n = L_{m+n} - (-1)^n L_{m-n}$$

to get:

$$L_{F_{i+4}+L_{i+1}} - (-1)^{L_{i+1}} L_{F_{i+4}-L_{i+1}} - (-1)^{F_i}[L_{F_{i+2}+L_{i+2}} - (-1)^{L_{i+2}} L_{F_{i+2}-L_{i+2}}] - L_{F_i+L_{i+3}} + (-1)^{L_{i+3}} L_{F_i - L_{i+3}} = 0.$$

This identity can be shown to be true because, of the six terms, it can be grouped into pairs of terms that cancel. Specifically,

(2) $L_{F_{i+4}+L_{i+1}} = L_{F_i+L_{i+3}}$

(3) $(-1)^{L_{i+1}} L_{F_{i+4}-L_{i+1}} = (-1)^{F_i}(-1)^{L_{i+2}} L_{F_{i+2}-L_{i+2}}$

(4) $(-1)^{F_i} L_{F_{i+2}+L_{i+2}} = (-1)^{L_{i+3}} L_{F_i - L_{i+3}}$

Equation (2) follows from the identity

$$F_{i+4} + L_{i+1} = F_i + L_{i+3},$$

which is straightforward to prove.

To prove equation (3), we use the fact that $L_{-n} = (-1)^n L_n$, so that

$$L_{F_{i+2}-L_{i+2}} = L_{-F_{i+2}+L_{i+2}}$$

since a simple parity argument shows that $F_{i+2} - L_{i+2}$ is always even. Then we note that $F_i + L_{i+2} \equiv L_{i+1} \pmod 2$, which also follows from a simple parity argument. Thus,

$$(-1)^{L_{i+1}} = (-1)^{F_i + L_{i+2}}$$

and we see that equation (3) is equivalent to

$$F_{i+4} - L_{i+1} = -F_{i+2} + L_{i+2},$$

which we again leave as a simple exercise for the reader.

For equation (4), we have similarly that $F_i \equiv L_{i+3} \pmod 2$, and hence equation (4) is equivalent to the easily proven

$$F_{i+2} + L_{i+2} = -F_i + L_{i+3},$$

where again we note that $F_i - L_{i+3}$ is always even.

Finally, we note a second identity analogous to (1):

(5) $F_{F_{i+1}} F_{L_{i+1}} - (-1)^{F_i} F_{F_{i-1}} F_{F_{i+2}} = F_{F_i} F_{F_{i+3}}$

whose proof is similar and is omitted.

Equations (1) and (5) appear to generate all the numerical examples I have found. If we let i have the forms $3k - 1$, $3k$, and $3k + 1$, we get the six identities:

$$F_{F_{3k+3}} F_{L_{3k}} + F_{F_{3k+1}} F_{L_{3k+1}} = F_{F_{3k-1}} F_{L_{3k+2}}$$
$$F_{F_{3k+4}} F_{L_{3k+1}} = F_{F_{3k+2}} F_{L_{3k+2}} + F_{F_{3k}} F_{L_{3k+3}}$$
$$F_{F_{3k+5}} F_{L_{3k+2}} + F_{F_{3k+3}} F_{L_{3k+3}} = F_{F_{3k+1}} F_{L_{3k+4}}$$
$$F_{F_{3k}} F_{L_{3k}} + F_{F_{3k-2}} F_{F_{3k+1}} = F_{F_{3k-1}} F_{F_{3k+2}}$$
$$F_{F_{3k+1}} F_{L_{3k+1}} = F_{F_{3k-1}} F_{F_{3k+2}} + F_{F_{3k}} F_{F_{3k+3}}$$
$$F_{F_{3k+2}} F_{L_{3k+2}} + F_{F_{3k}} F_{F_{3k+3}} = F_{F_{3k+1}} F_{F_{3k+4}}$$

which are probably the ones the proposer had in mind.

Also solved by P. Bruckman and the proposer.

Squared Magic

H-455 *Proposed by T. V. Padma Kumar, Trivandrum, South India (Vol. 29, no. 3, August 1991)*

Characterize, as completely as possible, all "Magic Squares" of the form

a_1	a_2	a_3	a_4
b_1	b_2	b_3	b_4
c_1	c_2	c_3	c_4
d_1	d_2	d_3	d_4

subject to the following constraints:

1. Rows, columns, and diagonals have the same sum
2. $a_1 + a_4 + d_1 + d_4 = b_2 + b_3 + c_2 + c_3 = a_1 + b_1 + a_4 + b_4 = K$
3. $c_1 + d_1 + c_4 + d_4 = a_2 + a_3 + b_2 + b_3 = c_2 + c_3 + d_2 + d_3 = K$
4. $a_1 + a_2 + b_1 + b_2 = c_1 + c_2 + d_1 + d_2 = a_3 + a_4 + b_3 + b_4 = K$
5. $c_3 + c_4 + d_3 + d_4 = c_1 + d_2 + a_3 + b_4 = a_1 + a_2 + d_1 + d_2 = K$
6. $a_3 + a_4 + d_3 + d_4 = b_1 + b_2 + c_1 + c_2 = b_3 + b_4 + c_3 + c_4 = K$
7. $a_2 + a_3 + d_2 + d_3 = b_1 + c_1 + b_4 + c_4 = K$
8. $a_1 + b_1 + c_1 + a_2 + b_2 + a_3 = b_4 + c_3 + c_4 + d_2 + d_3 + d_4 = 3K/2$
9. $b_1 + c_1 + d_1 + c_2 + d_2 + d_3 = a_2 + a_3 + a_4 + b_3 + b_4 + c_4 = 3K/2$
10. $a_2^2 + a_3^2 + d_2^2 + d_3^2 = b_1^2 + c_1^2 + b_4^2 + c_4^2$
11. $c_1^2 + c_2^2 + d_1^2 + d_2^2 = a_3^2 + b_3^2 + a_4^2 + b_4^2$
12. $c_3^2 + c_4^2 + d_3^2 + d_4^2 = a_1^2 + b_1^2 + a_2^2 + b_2^2$
13. $a_1^2 + a_2^2 + a_3^2 + a_4^2 + b_1^2 + b_2^2 + b_3^2 + b_4^2 = M$
14. $c_1^2 + c_2^2 + c_3^2 + c_4^2 + d_1^2 + d_2^2 + d_3^2 + d_4^2 = M$
15. $a_1^2 + b_1^2 + c_1^2 + d_1^2 + a_2^2 + b_2^2 + c_2^2 + d_2^2 = M$
16. $a_3^2 + b_3^2 + c_3^2 + d_3^2 + a_4^2 + b_4^2 + c_4^2 + d_4^2 = M$
17. $a_1 + b_2 + c_3 + d_4 + d_1 + c_2 + b_3 + a_4 = b_1 + c_1 + a_2 + d_2 + a_3 + d_3 + b_4 + c_4$
18. $a_1a_2 + a_3a_4 + b_1b_2 + b_3b_4 = c_1c_2 + c_3c_4 + d_1d_2 + d_3d_4$
19. $a_1b_1 + c_1d_1 + a_2b_2 + c_2d_2 = a_3b_3 + c_3d_3 + a_4b_4 + c_4d_4$

Solution by Paul S. Bruckman, Edmonds, WA

We first apply constraints 1-9 and 17, which are linear in nature. We find that these constraints are satisfied with 4 degrees of freedom, that is, with 4 of the 16 unknown quantities still undetermined. We may choose any 4 of the 16 quantities as arbitrary and determine the other 12 from these, so as to satisfy 1-9 and 17. For example, if we leave a_1, a_2, a_3, and b_1 as arbitrary, our magic square will look as follows:

a_1	a_2	a_3	$k - a_1 - a_2 - a_3$
b_1	$k - a_1 - a_2 - b_1$	$a_1 + b_1 - a_3$	$a_2 + a_3 - b_1$
$\frac{k}{2} - a_3$	$a_1 + a_2 + a_3 - \frac{k}{2}$	$\frac{k}{2} - a_1$	$\frac{k}{2} - a_2$
$\frac{k}{2} - a_1 - b_1 + a_3$	$\frac{k}{2} - a_2 - a_3 + b_1$	$\frac{k}{2} - b_1$	$a_1 + a_2 + b_1 - \frac{k}{2}$

It is a tedious but trivial exercise to verify that the quantities shown above satisfy constraints 1-9 and 17, and also constraints 10-12, 18, and 19. As for constraints 13-16, we may also verify that these are satisfied by the above quantities, provided the following single condition holds:

$$(*) \qquad M = 2k^2 - 2k(2a_1 + 2a_2 + a_3 + b_1) + 4b_1^2 + 4b_1(a_1 - a_3) + 4a_2(a_1 + a_3) + 4(a_1^2 + a_2^2 + a_3^2).$$

The condition in (*) removes one additional degree of freedom, thereby leaving only 3 undetermined quantities, say a_1, a_2, and a_3. If we require that the magic square's entries be *integers*, this imposes additional constraints on the entries, subject to the Diophantine solutions of (*). If, in addition, we require that the entries be *distinct*, further restrictions apply.

As may be shown, the corner entries of any 3×3 square contained within the large square must add up to k, as well as the corner entries of the large square itself. Moreover, the entries of any 2×2 square contained within the large square must total k.

An example which satisfies all 19 conditions (though not the condition that the entries be distinct) is the following, taking $k = 18$, $M = 208$, $a_1 = 4$, $a_2 = 3$, and $a_3 = 5$:

4	3	5	6
2	9	1	6
4	3	5	6
8	3	7	0

If we take $k = 34$, $M = 748$, $a_1 = 5$, $a_2 = 11$, $a_3 = 8$, we obtain a "conventional" magic square (where all entries are integers; in fact, the integers from 1-16). There are many such magic squares possible; this is only one such:

5	11	8	10
16	2	13	3
9	7	12	6
4	14	1	15

Also solved by the proposer.

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