A GENERALIZATION OF KUMMER'S CONGRUENCES AND RELATED RESULTS

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(Submitted February 1991)

1. Introduction

Euler's ϕ-function $\phi(m)$ for m a natural number is defined to be the number of natural numbers not exceeding m which are relatively prime to m. Euler's Theorem states: If m is a natural number and a is an integer such that $(a, m) = 1$, then $a^{\phi(m)} \equiv 1 \pmod{m}$. It is well known that if $m > 1$ and

$$m = p_1^{a_1} p_2^{a_2} \ldots p_t^{a_t}$$

is m's unique representation as a product of pairwise distinct prime numbers, then

$$\phi(m) = m\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \ldots \left(1 - \frac{1}{p_t}\right).$$

For a discussion of Euler's ϕ-function, see [19], pages 180-83 and 185-90. For clarity of notation,

$$\mathrm{GCD}(a, b) = (a, b)$$

occasionally will be used for the greatest common divisor of a and b. Also,

$$\mathrm{LCM}[a_1, a_2, \ldots, a_t]$$

will be used for the least common multiple of $a_1, a_2, \ldots, a_t$. As will be seen, the ϕ-function is useful for generating sequences of rational numbers which are used to construct generalized Kummer congruences.

This paper is concerned with sequences $\{u_j\}_{j=0}^{\infty}$ of rational numbers. It will be supposed that each such rational number is written as a quotient of relatively prime integers. A rational number so written is said to be in *standard form*. It is immaterial for this discussion whether the denominator be positive or negative.

The purpose of this paper is to develop a method which will generate sequences of rational numbers (e_n-sequences) which satisfy Kummer's congruence (see line 9 in Definition 3) and especially Theorem 7. The sequences are manifold: they include Bernoulli, Euler, and Tangent numbers as well as Bernoulli and Euler polynomials. Some additional applications will also be given. For example, Kummer's congruences involving reciprocals of Bernoulli (Theorem 9) and Euler numbers (Theorem 8) will be given. A ring structure for some of these sequences will be observed (section 7), and finally some additional examples will be given (section 8).

The Bernoulli polynomials $\{B_j(x)\}_{j=0}^{\infty}$ are defined by

$$(1) \qquad \frac{te^{xt}}{e^t - 1} = \sum_{j=0}^{\infty} B_j(x)\frac{t^j}{j!},$$

and the Bernoulli numbers $\{B_j\}_{j=0}^{\infty}$ are defined by the generating function

$$(2) \qquad \frac{x}{e^x - 1} = \sum_{j=0}^{\infty} B_j \frac{x^j}{j!}.$$

See [21], pages 167 and 35.

A rational number a in standard form is a *p-integer* for the prime number p provided the denominator of a is relatively prime to p. See [1], pages 22 and 385. *Kummer's congruence* says: If p is a prime number and $k \not\equiv 0 \pmod{p-1}$ where k is an even natural number, then B_k/k is a p-integer and

$$(3) \qquad \frac{B_{k+p-1}}{k+p-1} \equiv \frac{B_k}{k} \pmod{p}.$$

In the paper [11] Fermat's Little Theorem was generalized to sequences $\{u_j\}_{j=0}^{\infty}$ of rational numbers which include sequences of the form $\{a^j\}_{j=0}^{\infty}$ where a is a rational. Basically, [11] investigated sequences $\{u_j\}_{j=0}^{\infty}$ having the property $u_p \equiv u_1 \pmod{p}$ for p a prime number. It is to be observed that $u_p \equiv u_1 \pmod{p}$ can be formed umbrally from $a^p \equiv a \pmod{p}$ by identifying superscripts with subscripts and changing a to u. Here congruences $(\text{mod } m^n)$ are investigated with $m > 1$ a natural number.

Definition 1: Let $m > 1$ be a natural number and let a be a rational number in standard form. The rational number a is said to be an *m-integer* or to be *m-integral* provided the denominator of a is relatively prime to m. If m is a prime number, then of course a is simply a p-integer.

The main results of this paper follow Theorem 1. However, Theorem 1 is important for Definition 3. See the remarks immediately following Definition 3.

Definition 2: Let $m > 1$ be a natural number and suppose $m = p_1^{a_1}p_2^{a_2} \ldots p_t^{a_t}$ is its unique representation as a produce of pairwise distinct prime numbers. The *height h(m) of m* is defined to be

$$(4) \qquad h(m) = \max_{1 \le j \le t} (a_j).$$

If $m = 1$, then $h(m)$ is defined to be 0.

Theorem 1 follows from results in [9] or can be easily proved directly.

Theorem 1: Let $m > 1$ be a natural number and suppose a is an m-integer. Then

$$(5) \qquad a^{\phi(m)+h(m)} - a^{h(m)} \equiv 0 \pmod{m}.$$

If $m = p$ a prime number, then

$$h(m) = h(p) = 1 \quad \text{and} \quad \phi(m) = \phi(p) = p - 1$$

so that Theorem 1 says $a^p - a \equiv 0 \pmod{p}$, which is Fermat's Theorem. If $(a, m) = 1$, then Theorem 1 is Euler's Theorem.

Using Euler's Theorem, if a is an m-integer, r an integer, g a natural number, and if r is negative $1/a$ is also an m-integer, Theorem 1 and induction give

$$(6) \qquad a^{r[\phi(m)+h(m)]^g} - a^{r[h(m)]^g} \equiv 0 \pmod{m}.$$

To see this, note that a^r is an m-integer whether r is positive or negative.

From (6) for n a natural number with r and k integers,

$$(7) \qquad a^k\left(a^{r[\phi(m)+h(m)]^g} - a^{r[h(m)]^g}\right)^n \equiv 0 \pmod{m^n}.$$

(See the second paragraph after Definition 4.)

Here, a and $1/a$ are both m-integers if either k or r is negative. This says that

$$(8) \qquad \sum_{j=0}^{n} (-1)^j \binom{n}{j} a^{(n-j)r[\phi(m)+h(m)]^g + r[h(m)]^g j + k} \equiv 0 \pmod{m^n}.$$

Viewing (8) umbrally gives the inspiration for the following Definition.

Definition 3: Let $m = \text{LCM}[m_1, m_2, \ldots, m_t] > 1$ where $m_1, m_2, \ldots, m_t$ are natural numbers. The sequence $\{u_j\}_{j=0}^{\infty}$ of rational numbers written in standard form such that each element of

$$\left\{u_{(n-j)\alpha(m)+\beta(m)j+\gamma(m)}\right\}_{j=0}^{n}$$

is an m-integer where $\alpha(m)$, $\beta(m)$, and $\gamma(m)$ are integers such that

$$f(n, j) = (n - j)\alpha(m) + \beta(m)j + \gamma(m) \geq 0$$

is an e_n*-sequence with shift* $(\alpha(m), \beta(m), \gamma(m))$ *with respect to m provided*

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}u_{f(n,j)} \equiv 0 \pmod{m_1^{n_1}m_2^{n_2}\ldots m_t^{n_t}}, \tag{9}$$

where $n_1, n_2, \ldots, n$ are whole numbers such that $n_1 + n_2 + \cdots + n_t = n$. This is, of course, equivalent to

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}u_{f(n,n-j)} \equiv 0 \pmod{m_1^{n_1}m_2^{n_2}\ldots m_t^{n_t}}.$$

In other words, $n_1, n_2, \ldots, n_t$ forms a whole number partition of the natural number n. (See the comments immediately following Theorem 8 and Definition 4.) It is easy to see that (9) can be replaced with the modulus

$$\{\text{LCM}[m_1, m_2, \ldots, m_t]\}^n.$$

(See the third paragraph below.) It is this form of (9) that will be used.

To say, for two rational numbers a and b, that $a \equiv b \pmod{m}$ for $m > 1$ a natural number simply means $(a - b)/m$ is an m-integer.

Theorem 1 does, as seen above, generalize Euler's Theorem. However, Theorem 1 is not the main generalization with which this paper is concerned. A sequence that is an e_n-sequence with shift $(\alpha(m), \beta(m), \gamma(m))$ could be called a *generalized Euler sequence*. Thus, this paper is not so much concerned with congruences of the form $a^{r+s} \equiv a^r \pmod{m}$ (see [5], [7], [9], [15]) as it is with sequences that satisfy (9). Kummer's congruences are related to congruences of the type (9) with the modulus

$$\{\text{LCM}[m_1, m_2, \ldots, m_t]\}^n = m^n.$$

Because of the special role that Euler's ϕ-function plays in finding many such congruences, it seems appropriate to refer to sequences named by Definition 3 as generalized Euler sequences.

In light of (8), one possible choice for $\alpha(m)$ and $\beta(m)$ is

$$\alpha(m) = r\alpha_1(m) \quad\text{and}\quad \beta(m) = r\beta_1(m)$$

where r is an integer and $\alpha_1(m)$ and $\beta_1(m)$ are such that, for some integers $r_1, r_2, \ldots, r_t$; $s_1, s_2, \ldots, s_t$ and some natural numbers $g_1, g_2, \ldots, g_t$;

$$\begin{aligned} r_1[\phi(m_1) + h(m_1)]^{g_1} + s_1 &= r_2[\phi(m_2) + h(m_2)]^{g_2} + s_2 \\ &= \cdots = r_t[\phi(m_t) + h(m_t)]^{g_t} + s_t = \alpha_1(m) \end{aligned}$$

and

$$\begin{aligned} r_1[h(m_1)]^{g_1} + s_1 &= r_2[h(m_2)]^{g_2} + s_2 \\ &= \cdots = r_t[h(m_t)]^{g_t} + s_t = \beta_1(m). \end{aligned}$$

To keep this shift from being trivial, $\alpha_1(m)$, $\beta_1(m)$, $r \neq 0$, and $\alpha_1(m) \neq \beta_1(m)$. This shift $(\alpha(m), \beta(m), \gamma(m))$ is a *natural shift*. It is clear that for a natural shift

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}a^{f(n,j)} \equiv 0 \pmod{m_1^{n_1}m_2^{n_2}\ldots m_t^{n_t}} \quad\text{for an } m\text{-integer.}$$

The reason for this is

$$(a^{\alpha(m)} - a^{\beta(m)})^{n_i} \equiv 0 \pmod{m_i^{n_i}}$$

so that

$$\prod_{i=1}^{t} (a^{\alpha(m)} - a^{\beta(m)})^{n_i} = (a^{\alpha(m)} - a^{\beta(m)})^{n_1+n_2+\cdots+n_t}$$

$$= \sum_{j=0}^{n} (-1)^j \binom{n}{j} a^{f(n,j)} \equiv 0 \pmod{m_1^{n_1} m_2^{n_2} \ldots m_t^{n_t}},$$

where $n_1, n_2, \ldots, n_t$ are whole numbers such that $n_1 + n_2 + \cdots + n_t = n$. Note that $\alpha(m)$ and $\beta(m)$ depend upon $m_1, m_2, \ldots, m_t$; $r_1, r_2, \ldots, r_t$; $s_1, s_2, \ldots, s_t$; and $g_1, g_2, \ldots, g_t$. Special care is needed when any of the r's or s's are negative. Note also, since the expression is divisible by $m_1^{n_1} m_2^{n_2} \ldots m_t^{n_t}$ for any whole number partition of $n = n_1 + n_2 + \cdots + n_t$, it will be divisible by $[\mathrm{LCM}[m_1, m_2, \ldots, m_t]]^n$ so that $\{u_j\}_{j=0}^{\infty}$ being an e_n-sequence with respect to $m = \mathrm{LCM}[m_1, m_2, \ldots, m_t]$ implies that

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} u_{f(n,j)} \equiv 0 \pmod{\{\mathrm{LCM}[m_1, m_2, \ldots, m_t]\}^n}$$

and conversely. Thus, for each way of writing m as $\mathrm{LCM}[m_1, m_2, \ldots, m_t]$ there is the possibility of a separate congruence (mod m^n). The simplest way of satisfying this is, of course, $m = \mathrm{LCM}[m]$. From now on, m will denote $\mathrm{LCM}[m_1, m_2, \ldots, m_t]$ for some natural numbers $m_1, m_2, \ldots, m_t$. As will be seen, other ways of writing m besides $m = \mathrm{LCM}[m]$ do indeed lead to different expressions $\equiv 0 \pmod{m^n}$. See section 8 for some examples. $[m_1, m_2, \ldots, m_t]$ is called an *LCM-partition of* m when $m = \mathrm{LCM}[m_1, m_2, \ldots, m_t]$ and $m_1, m_2, \ldots, m_t$ are all natural numbers > 1.

Definition 4: Let $\{u_j\}_{j=0}^{\infty}$ be a sequence of rational numbers written in standard form such that each element of $\{u_{(n-j)\alpha(m)+\beta(m)j+\gamma(m)}\}_{j=0}^{\infty}$ is an m-integer where $\alpha(m)$, $\beta(m)$, and $\gamma(m)$ are integers such that

$$f(n, j) = (n - j)\alpha(m) + \beta(m)j + \gamma(m) \geq 0.$$

If

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} u_{f(n,j)} \equiv 0 \pmod{m^n}, \text{ where } m = \mathrm{LCM}[m_1, m_2, \ldots, m_t] > 1$$

for some natural numbers $m_1, m_2, \ldots, m_t$, then this congruence is a *generalized Kummer congruence*.

From the above, if $\{u_j\}_{j=0}^{\infty}$ is an e_n-sequence with shift $(\alpha(m), \beta(m), \gamma(m))$ with respect to m, then it satisfies a generalized Kummer congruence.

A remark on $\phi(m)$ and $h(m)$ is needed: these functions are convenient to use; however, if for some natural number $m > 1$ there exist $A(m)$ and $B(m)$ such that for every m-integer a, $a^{A(m)} - a^{B(m)} \equiv 0 \pmod{m}$, then $A(m)$ could be used in place of $\phi(m) + h(m)$, and $B(m)$ in place of $h(m)$. Consequently, many of the results in this paper can be generalized somewhat by just such a consideration. However, because of the convenience of finding and working with $\phi(m)$ and $h(m)$, the results are stated in terms of these two functions. Furthermore, some of the parity properties of $\phi(m)$ are used in the proof of Theorem 2, so it was felt that it was better to state the results in terms of natural shifts.

There exist sequences $\{u_j\}_{j=0}^{\infty}$ with shifts other than the natural shift

$$(r[\phi(m) + h(m)]^g, r[h(m)]^g, \gamma(m)).$$

For example, using Theorem 5, if p is an odd prime and a is a p-integer such that

$$(a, p) = 1 \quad \text{and} \quad \left\{\frac{1}{(i - j)a^p + aj}\right\}_{j=0}^{n} \text{ for } 1 \leq i \leq n$$

are all p-integers, then the sequence $\{1/j\}_{j=1}^{\infty}$ is an e_n-sequence with shift $(a^p, a, 0)$ with respect to p. The condition

$$\frac{1}{(i-j)a^p + aj} \text{ is a } p\text{-integer for } 1 \le i \le n$$

is equivalent to $p > n$. Thus $\{1/j\}_{j=1}^{\infty}$ is an e_n-sequence with shift $(a^p, a, 0)$ when $p > n$. Here $m = \text{LCM}[p]$.

From the above definition, it is clear that linear combinations of e_n-sequences with common shift $(\alpha(m), \beta(m), \gamma(m))$ with respect to the same natural number $m > 1$ are also e_n-sequences with shift $(\alpha(m), \beta(m), \gamma(m))$ when the coefficients defining the linear combinations are all m-integers. In particular, multiplying each term of an e_n-sequence by an m-integer gives an e_n-sequence.

It is possible to couch condition (9) in terms of the difference operator Δ, here defined by $\Delta u_x = u_{x+t} - u_x$. If

$$x = n\beta(m) + \gamma(m) \quad \text{and} \quad t = \alpha(m) - \beta(m),$$

then it turns out that

$$\Delta^n u_x = \sum_{j=0}^{n} (-1)^j \binom{n}{j} u_{f(n,j)}.$$

Note that if

$$\alpha(m) = \phi(m) + h(m) \quad \text{and} \quad \beta(m) = h(m),$$

then the increment t is just $\phi(m)$. This will be returned to later in connection with the Factor and Product Theorems.

Let $\{L_j\}_{j=0}^{\infty}$ be the sequence of Lucas numbers. It is well known that

$$L_j = \left(\frac{1+\sqrt{5}}{2}\right)^j + \left(\frac{1-\sqrt{5}}{2}\right)^j, \quad j \ge 0. \quad \text{(See [13], page 26.)} \tag{10}$$

Although (10) represents L_j in the form $\alpha^j + \beta^j$, neither α nor β is rational. By the main theorem of [11], $\{L_j\}_{j=0}^{\infty}$ is an e_1-sequence for any prime number p with shift $(p, 1, 0)$; i.e., for p a prime number, $L_p \equiv L_1 \pmod{p}$. However, simply because L_j is the sum of powers of $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$, this is not sufficient for $\{L_j\}_{j=0}^{\infty}$ to be an e_n-sequence with arbitrary shift. Indeed, $\{L_j\}_{j=0}^{\infty}$ is not even an e_2-sequence with shift $(p, 1, 0)$ for the prime number $p = 3$ since $L_6 - 2L_4 + L_2 \not\equiv 0 \pmod{3^2}$. Hence, it does not follow that if each term of the sequence $\{u\ \}_{j=0}^{\infty}$ of rationals is of the form

$$u_j = x_1^j + x_2^j + \cdots + x_t^j$$

then the sequence is an e_n-sequence with even reasonable shifts.

2. Euler Polynomials and Numbers

The Euler polynomials $E_n(x)$ of degree n and argument x are given by the generating function

$$\frac{2e^{xt}}{1+e^t} = \sum_{j=0}^{\infty} \frac{E_j(x)t^j}{j!}. \quad \text{(See [21], page 175.)} \tag{11}$$

A well-known formula involving the Euler polynomials is

$$\sum_{i=1}^{N} (-1)^{N-i} i^n = \frac{1}{2}\{E_n(N+1) + (-1)^N E_n(0)\}, \tag{12}$$

where $n = 1, 2, 3, \ldots$, and $N = 1, 2, 3, \ldots$. (See [16], page 30.)

Using the notation introduced in Definition 3, replace n by $f_j = f(n, j)$ in (12) so that

(13) $$\sum_{i=1}^{N}(-1)^{N-i}i^{f_j} = \frac{1}{2}\{E_{f_j}(N+1) + (-1)^N E_{f_j}(0)\}.$$

To (13), apply the operator

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}a^{f_j}$$

so that

(14) $$\sum_{j=0}^{n}(-1)^j\binom{n}{j}\sum_{i=1}^{N}(-1)^{N-i}a^{f_j}i^{f_j}$$

$$= \frac{1}{2}\sum_{j=0}^{n}(-1)^j\binom{n}{j}a^{f_j}E_{f_j}(N+1) + \frac{1}{2}(-1)^N\sum_{j=0}^{n}(-1)^j\binom{n}{j}a^{f_j}E_{f_j}(0).$$

Expanding the left side of (14) gives

(15) $$\sum_{j=0}^{n}(-1)^j\binom{n}{j}(-1)^{N-1}[a^{f_j} - (2a)^{f_j} + (3a)^{f_j} - + \cdots + (-1)^{N-1}(Na)^{f_j}]$$

$$= (-1)^{N-1}\{a^{\gamma(m)}[a^{\alpha(m)} - a^{\beta(m)}]^n - (2a)^{\gamma(m)}[(2a)^{\alpha(m)} - (2a)^{\beta(m)}]^n$$

$$+ - \cdots + (-1)^{N-1}(Na)^{\gamma(m)}](Na)^{\alpha(m)} - (Na)^{\beta(m)}]^n\}.$$

Now if $\alpha(m)$ and $\beta(m)$ are such that

$$[(ia)^{\alpha(m)} - (ia)^{\beta(m)}]^n \equiv 0 \pmod{m^n} \text{ for } i = 1, 2, \ldots, N$$

where $m = \text{LCM}[m_1, m_2, \ldots, m_t]$, which they will be for the natural shift $(\alpha(m), \beta(m), \gamma(m))$, then by (7) for $a^{\gamma(m)}$, $(ia)^{\alpha(m)}$, $(ia)^{\beta(m)}$ all m-integral for $i = 1, 2, 3, \ldots, N$, (15) will be $\equiv 0 \pmod{m^n}$. Because of the conditions needed for all these numbers to be m-integers, it is supposed that $r \geq 0$ and $\gamma(m) \geq 0$.

Suppose that $\alpha(m) = r\alpha_1(m)$ and $\beta(m) = r\beta_1(m)$. For $m_i = 2$ where $i = 1, 2, \ldots, t$, the parity of $f(n, j)$ is the parity of $rr_1j + \gamma + nrs_1$, which will be even if r and $\gamma(m)$ are both even. On the other hand, if $m_i > 2$ for some $i = 1, 2, \ldots, t$, all of the numbers $f(n, j)$, $0 \leq j \leq n$, have the same parity. To see this, use the fact that $\phi(m_i)$ is even when $m_i > 2$. From (15) and (14),

(16) $$\frac{1}{2}\sum_{j=0}^{n}(-1)^i\binom{n}{j}a^{f(n,j)}E_{f(n,j)}(N+1) + \frac{1}{2}(-1)^N\sum_{j=0}^{n}(-1)^j\binom{n}{j}a^{f(n,j)}E_{f(n,j)}(0)$$
$$\equiv 0 \pmod{m^n}.$$

It is well known that, for $f(n, j)$ even, $E_{f(n,j)}(0) = 0$. (See [21], page 179.) Now $f(n, j)$ is even when $\beta_1(m)$ is odd and $nr + \gamma(m)$ is even when $\beta_1(m)$ is even and $\gamma(m)$ is even.

Next, suppose that m is odd so that $1/2$ is m-integral. In this case, for $N \equiv -(1/2) \pmod{m^n}$ and $f(n, j)$ odd, then

$$E_{f(n,j)}\left(\frac{1}{2}\right) = 0,$$

whereas, if $f(n, j)$ is even

$$E_{f(n,j)}(1) = 0 \quad [\text{letting } N \equiv 0 \pmod{m^n}]. \quad \text{(See [21], page 179.)}$$

Hence, in (14),

$$\sum_{j=0}^{n}(-1)^i\binom{n}{j}E_{f(n,j)}(0) \equiv 0 \pmod{m^n}$$

when $f(n, j)$ is even or when m is odd. Since n is a natural number in (12) and $f(n, j)$ replaces n, it follows that $f(n, j) \geq 1$. This establishes the following theorem.

Theorem 2: Let $m = \text{LCM}[m_1, m_2, \ldots, m_t] > 1$ with $m_1, m_2, \ldots, m_t$ all natural numbers and a, $\gamma(m)$, and x all m-integers. Suppose

$$f(n, j) = (n - j)r\alpha_1(m) + r\beta_1(m)j + \gamma(m) \geq 1 \quad \text{for } 0 \leq j \leq n,$$

where $r \geq 0$ and $\gamma(m) \geq 0$. Assume one of the following statements holds:

(1) $m_i = 2$ for $i = 1, 2, \ldots, t$, and r and $\gamma(m)$ are both even;
(2) m is even and $m_i > 2$ for some $i = 1, 2, \ldots, t$, $\beta_1(m)$ and $\gamma(m)$ are both even;
(3) m is even and $m_i > 2$ for some $i = 1, 2, \ldots, t$, and $nr + \gamma(m)$ is even but $\beta_1(m)$ is odd;
(4) m is odd.

Then $\{a^{f(n,j)}E_{f(n,j)}(x)\}_{j=0}^{n}$ are all m-integers and $\{a^j E_j(x)\}_{j=0}^{\infty}$ is an e_n-sequence with the natural shift $(\alpha(m), \beta(m), \gamma(m))$.

The hypothesis of Theorem 2 cannot be weakened to simply: $m > 1$ is a natural number. To see this, let $m = 4 = \text{LCM}[4]$, $n = 1$, $g = r = \gamma(m) = 1$. None of the four hypotheses is satisfied if $r_1 = 1$ and $s_1 = 0$. If the weakened hypothesis is valid, then

$$(17) \qquad \sum_{j=0}^{1}(-1)^j\binom{1}{j}E_{5-2j}(x) = E_5(x) - E_3(x)$$
$$= \left(x^5 - \frac{5x^4}{2} + \frac{5x^2}{2} - \frac{1}{2}\right) - \left(x^3 - \frac{3x^2}{2} + \frac{1}{4}\right) \equiv 0 \pmod 4$$

which is false.

For $m > 2$ and m odd, the coefficients of the Euler polynomials are all m-integers. To see this, use

$$(18) \qquad E_n(x) = 2^{-n}\sum_{j=0}^{n}\binom{n}{j}(2x - 1)^{n-j}E_j,$$

where $\{E_j\}_{j=0}^{\infty}$ is the sequence of Euler numbers. The Euler numbers are all integers and, furthermore, $E_t = 2^t E_t(1/2)$. (See [21], pages 177, 39, and 42.)

The above observations along with Theorem 2 establish Theorem 3.

Theorem 3: Let $m = \text{LCM}[m_1, m_2, \ldots, m_t] > 1$ with $m_1, m_2, \ldots, m_t$ all natural numbers and a an m-integer. Suppose

$$f(n, j) = (n - j)r\alpha_1(m) + r\beta_1(m)\, j + \gamma(m) \geq 1 \quad \text{for } 0 \leq j \leq n,$$

where $r \geq 0$ and $\gamma(m) \geq 0$. Then $\{a^j E_j\}_{j=0}^{\infty}$ is an e_n-sequence with the natural shift $(\alpha(m), \beta(m), \gamma(m))$.

The Euler numbers form secant coefficients since

$$\sec x = \sum_{j=0}^{\infty}(-1)^j\frac{E_{2j}x^{2j}}{(2j)!},$$

which is convergent for $|x| < \pi/2$. The number $E_{2n+1} = 0$ for $n \geq 0$. (See [18], pages 202 and 203.)

3. Bernoulli Numbers and Polynomials

The above results open the way to exploration of Bernoulli polynomials and Bernoulli numbers with respect to forming e_n-sequences. A useful relationship is

$$(19) \qquad E_n(x) = \frac{2^{n+1}}{n+1}\left[B_{n+1}\left(\frac{x+1}{2}\right) - B_{n+1}\left(\frac{x}{2}\right)\right] \quad \text{for } n = 0, 1, 2, \ldots .$$

(See [21], page 177.) Using this and the hypothesis of Theorem 2, we have

$$(20) \qquad \left\{\frac{2^{j+1}a^j}{j+1}\left[B_{j+1}\left(\frac{x+1}{2}\right) - B_{j+1}\left(\frac{x}{2}\right)\right]\right\}_{j=0}^{\infty}$$

is an e_n-sequence with natural shift $(\alpha(m), \beta(m), \gamma(m))$ for the natural number $m = \mathrm{LCM}[m_1, m_2, \ldots, m_t] > 1$. Here, both a and x are m-integers.

In (20) let $x = 0$ so that

$$B_{j+1}\left(\frac{1}{2}\right) = \left(\frac{2}{2^{j+1}} - 1\right)B_{j+1} \quad \text{and} \quad B_{j+1}(0) = B_{j+1}, \text{ for } j = 1, 3, 5, \ldots .$$

(See [21], page 171.)

After simplification and using $B_{2j+1} = 0$ for $j = 1, 2, 3, \ldots$, (20) gives

Theorem 4: Let $m = \mathrm{LCM}[m_1, m_2, \ldots, m_t] > 1$ with $m_1, m_2, \ldots, m_t$ all natural numbers and let a be an m-integer. Suppose

$$f(n, j) = (n - j)\alpha(m) + \beta(m)j + \gamma(m) \geq 1 \quad \text{for } 0 \leq j \leq n,$$

where $r \geq 0$ and $\gamma(m) \geq 0$. If m is odd, then

$$\left\{\left(2^{f(n,j)+1} - 1\right)a^{f(n,j)+1}\frac{B_{f(n,j)+1}}{f(n, j) + 1}\right\}_{j=0}^{n}$$

are all m-integers and

$$\left\{\left(2^{j+1} - 1\right)a^{j+1}\frac{B_{j+1}}{j + 1}\right\}_{j=0}^{\infty} \tag{21}$$

is an e_n-sequence with the natural shift $(\alpha(m), \beta(m), \gamma(m))$.

It is important in working with these e_n-sequences to *first* put the terms in standard form and *then* reduce the expression (mod m^n).

Theorem 4 generalizes some well-known results. With the hypotheses of Theorem 4, (21) says

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{\left(2^{[r[\phi(m)+h(m)]^g - r[h(m)]^g]j+k} - 1\right)B_{[r[\phi(m)+h(m)]^g - r[h(m)]^g]j+k}}{[r[\phi(m) + h(m)]^g - r[h(m)]^g]j + k} \equiv 0 \pmod{m^n}, \tag{22}$$

where $k = r[\phi(m)]^g n + \gamma(m) + 1$. Here $m = \mathrm{LCM}[m]$. This last condition is equivalent to saying $k > r[\phi(m)]^g n$. If $m = p$ (a prime number), $r = g = 1$, then (22) gives

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{\left(2^{(p-1)j+k} - 1\right)B_{(p-1)j+k}}{(p - 1)j + k} \equiv 0 \pmod{p^n},\ k > (p - 1)n. \tag{23}$$

The Bernoulli, Genocchi, Lucas, and Euler numbers are closely related (see [14]). In particular,

$$G_n = 2(1 - 2^n)B_n \quad \text{and} \quad R_n = (1 - 2^{n-1})B_n, \tag{24}$$

where G_n and R_n are the Genocchi and Lucas numbers, respectively. With the same hypothesis as Theorem 4, $m = p = \mathrm{LCM}[p]$ and $r = g = 1$ gives as examples

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{G_{(p-1)j+k}}{(p - 1)j + k} \equiv 0 \pmod{p^n}, \text{ and} \tag{25}$$

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{(2^{(p-1)j+k} - 1)R_{(p-1)j+k}}{(1 - 2^{(p-1)j+k-1})((p - 1)j + k)} = 0 \pmod{p^n}.$$

For a further discussion of these numbers, see [6] and [25].

4. The Factor and Product Theorems

In (21) it is clear that $\{2^{j+1} - 1\}_{j=0}^{\infty}$ is an e_n-sequence with natural shift $(\alpha(m), \beta(m), \gamma(m))$ for the natural number $m = \mathrm{LCM}[m_1, m_2, \ldots, m_t]$. This suggests the possibility of "factoring" a sequence of the form $\{u_j v_j\}_{j=0}^{\infty}$. To that end, consider

(26) $$\Delta^n u_x v_x = \sum_{i=0}^{n} \binom{n}{i} (\Delta^{n-i} u_x)(\Delta^i v_{x+(n-i)t}),$$

where

(27) $$\Delta^n u_x = \sum_{j=0}^{n} (-1)^j \binom{n}{j} u_{x+(n-j)t}.$$

Here, the difference operator is defined by $\Delta u_x = u_{x+t} - u_x$. (See [10], pages 6 and 1, respectively.) Rewriting (26) using (27) gives

(28) $$\sum_{j=0}^{n} (-1)^j \binom{n}{j} u_{x+(n-j)t} v_{x+(n-j)t}$$
$$= \sum_{i=0}^{n} \binom{n}{i} \left(\sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} u_{x+(n-i-j)t} \right) \left(\sum_{j=0}^{i} (-1)^j \binom{i}{j} v_{x+(n-j)t} \right).$$

To express this in a form needed for e_i-sequences, let

(29) $$x + (n - j)t = (n - j)\alpha(m) + \beta(m)j + \gamma(m), \text{ so that}$$
$$x = n\beta(m) + \gamma(m) \quad \text{and} \quad t = \alpha(m) - \beta(m).$$

Substituting these in (28) yields

(30) $$\sum_{j=0}^{n} (-1)^j \binom{n}{j} u_{(n-j)\alpha(m)+\beta(m)j+\gamma(m)} v_{(n-j)\alpha(m)+\beta(m)j+\gamma(m)}$$
$$= \sum_{i=0}^{n} \left[\binom{n}{i} \left(\sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} u_{(n-i-j)\alpha(m)+\beta(m)j+\beta(m)i+\gamma(m)} \right) \cdot \left(\sum_{j=0}^{i} (-1)^j \binom{i}{j} v_{(n-j)\alpha(m)+\beta(m)j+\gamma(m)} \right) \right].$$

Using this, the Factor Theorem is obtained.

Theorem 5 (Factor Theorem): Let $m = \text{LCM}[m_1, m_2, \ldots, m_t]$ with $m_1, m_2, \ldots, m_t$ natural numbers. If

(a) $\{u_j v_j\}_{j=0}^{\infty}$ is an e_n-sequence with shift $(\alpha(m), \beta(m), \gamma(m))$; and

(b) $\{v_j\}_{j=0}^{\infty}$ is an e_i-sequence with shift $(\alpha(m), \beta(m), (n - i)\alpha(m) + \gamma(m))$, for $i = 1, 2, \ldots, n - 1$; and

(c) $\{u_j\}_{j=0}^{\infty}$ is an e_{n-i}-sequence with shift $(\alpha(m), \beta(m), \beta(m)i + \gamma(m))$ for $i = 1, 2, \ldots, n - 1$, then

1) If $(m, v_{n\alpha(m)+\gamma(m)}) = 1$ and $\{v_j\}_{j=0}^{\infty}$ is an e_n-sequence with shift $(\alpha(m), \beta(m), \gamma(m))$, then $\{u_j\}_{j=0}^{\infty}$ is an e_n-sequence with shift $(\alpha(m), \beta(m), \gamma(m))$;

2) If $(m, u_{n\beta(m)+\gamma(m)}) = 1$ and $\{u_j\}_{j=0}^{\infty}$ is an e_n-sequence with shift $(\alpha(m), \beta(m), \gamma(m))$, then $\{v_j\}_{j=0}^{\infty}$ is an e_n-sequence with shift $(\alpha(m), \beta(m), \gamma(m))$.

An examination of identity (30) also leads to the Product Theorem.

Theorem 6 (Product Theorem): Let $m = \text{LCM}[m_1, m_2, \ldots, m_t] > 1$ with $m_1, m_2, \ldots, m_t$ natural numbers. If

(a) $\{u_j\}_{j=0}^{\infty}$ is an e_{n-i}-sequence with shift $(\alpha(m), \beta(m), \beta(m)i + \gamma(m))$ for $i = 0, 1, 2, \ldots, n - 1$; and

(b) $\{v_j\}_{j=0}^{\infty}$ is an e_i-sequence with shift $(\alpha(m), \beta(m), (n - i)\alpha(m) + \gamma(m))$ for $i = 1, 2, \ldots, n$; thus, $\{u_j v_j\}_{j=0}^{\infty}$ is an e_n-sequence with shift $(\alpha(m), \beta(m), \gamma(m))$.

Using $m > 1$ being odd and $\gamma(m) \ge 0$ arbitrary, Theorem 4 together with the Factor Theorem and Theorem 1 yields

Theorem 7: Let $m = \mathrm{LCM}[m_1, m_2, \ldots, m_t] > 1$ with $m_1, m_2, \ldots, m_t$ all natural numbers. If

(a) $f(n, j) = (n - j)r\alpha_1(m) + r\beta_1(m) + \gamma(m)$ is an odd natural number for $0 \le j \le n$; and

(b) $r \ge 0$, $\gamma(m) \ge 0$, g is a natural number; and

(c) $\mathrm{GCD}\left(m,\ 2^{ir\alpha_1(m)+\gamma(m)+1} - 1\right) = 1$ or, equivalently
$\mathrm{GCD}\left(m,\ 2^{ir\beta_1(m)+\gamma(m)+1} - 1\right) = 1$ for $i = 1, 2, \ldots, n$,

then $\left\{\dfrac{B_{f(n,j)+1}}{f(n, j) + 1}\right\}_{j=0}^{n}$ are all m-integers and $\left\{\dfrac{B_{j+1}}{j + 1}\right\}_{j=0}^{\infty}$ is an e_n-sequence with the natural shift $(\alpha(m), \beta(m), \gamma(m))$.

In Theorem 7 let $m = p = \mathrm{LCM}[p]$ be an odd prime number and suppose $r = g = 1$ and $k = n + \gamma + 1$. Then (c) becomes

$$(p,\ 2^{i+k-n} - 1) = 1, \quad i = 1, 2, \ldots, n.$$

If $(p, 2^k - 1) = 1$, then $k \not\equiv 0 \pmod{p - 1}$ since $p \mid (2^{p-1} - 1)$ by Fermat's Little Theorem. Theorem 7 gives

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{B_{(p-1)j+k}}{(p - 1)j + k} \equiv 0 \pmod{p^n}. \tag{31}$$

This congruence is well known (see [3], [4], [18], [22], [23], [24], [26]). The paper [22] has many references to these and related congruences. It is clear that Theorem 7 with $m = p = \mathrm{LCM}[p]$ does not remove the restriction $k \not\equiv 0 \pmod{p - 1}$.

In Theorem 7 let $m = p^t$, where p is an odd prime number and t is a natural number. Then

$$\phi(m) = \phi(p^t) = p^{t-1}(p - 1) \quad \text{and} \quad h(m) = h(p^t) = t.$$

Further, suppose that $\gamma(m) = \gamma(p^t) \ge 0$, $r \ge 0$, g is a natural number and $n = 1$. Then Theorem 7 gives

$$\frac{B_{r[p^{t-1}(p-1)+t]^g+\gamma+1}}{r[p^{t-1}(p - 1) + t]^g + \gamma + 1} \equiv \frac{B_{rt^g+\gamma+1}}{rt^g + \gamma + 1} \pmod{p^t}, \tag{32}$$

when $(p, 2^{t+\gamma+1} - 1) = 1$. In (32) let $t = 1$ and $\gamma = 2k - 2$. This then is Kummer's congruence with the hypothesis $(p, 2^{2k} - 1) = 1$. Similar congruences immediately follow from Theorem 7 for $m = p^t$ and n an arbitrary natural number.

Repeated use of the Product Theorem allows for variations of the previous results. Thus, for $m > 1$ an odd natural number $\{\alpha^j E_{j+b_1}^{a_1} E_{j+b_2}^{a_2} \cdots E_{j+b_t}^{a_t}\}_{j=0}^{\infty}$ is an e_n-sequence with shift $(r[\phi(m) + h(m)]^g, r[h(m)]^g, \gamma(m))$ where $r \ge 0$, $\gamma(m) \ge 0$, $a_1, a_2, \ldots, a_t$; $b_1, b_2, \ldots, b_t$ are whole numbers and α is an m-integer. One application of this is to let

$$a_1 = a_2 = \cdots = a_t = 1 \quad \text{and} \quad b_1 = b_2 = \cdots = b_t = 0$$

so that $\{E_j^t\}_{j=0}^{\infty}$ is an e_n-sequence. For example, let $m = p = \mathrm{LCM}[p]$ be an odd prime number and let $n = 2$. Then, for t any natural number,

$$E_{2p+\gamma}^t - 2E_{p+\gamma+1}^t + E_{\gamma+2}^t \equiv 0 \pmod{p^2}.$$

Here, $\gamma = \gamma(p) \ge 1$ and $r = 1$. For example, letting $p = 7$ and $\gamma = 2$, this says, after reduction, for every n a whole number

$$40^n - 2 \cdot 47^n + 5^n \equiv 0 \pmod{49}.$$

It is possible to combine both the Factor Theorem and the Product Theorem. Since $\{1\}_{j=0}^{\infty}$ is an e_n-sequence with respect to the odd natural number $m > 1$ and for j even, $E_j(1/E_j) = 1$, it follows that for the natural shift with $r \geq 0$, $\gamma(m) \geq 0$, and $f(n, j)$ even, for $0 \leq j \leq n$ and $\{1/E_{f(n,j)}\}_{j=0}^{n}$ consisting of m-integers, then $\{1/E_j\}_{j \text{ even}}$, is an e_n-sequence. From Theorem 3 it follows that

$$E_{f(n,j+1)} \equiv E_{f(n,j)} \pmod{m} \text{ for } 0 \leq j+1 \leq n,$$

so that if $(m, E_{f(n,j)}) = 1$ for any $j = 0, 1, 2, \ldots, n$, then $\{1/E_{f(n,j)}\}_{j=0}^{n}$ consists of m-integers. This establishes

Theorem 8: Let $m = \text{LCM}[m_1, m_2, \ldots, m_t] > 1$ with $m_1, m_2, \ldots, m_t$ natural numbers be an odd natural number. Suppose

$$f(n, j) = (n - j)r\alpha_1(m) + r\beta_1(m) + \gamma(m)$$

is an even natural number where $r \geq 0$ and $\gamma(m) \geq 0$. If $(m, E_{f(n,j)}) = 1$ for at least one $j = 0, 1, 2, \ldots, n$, then the sequence $\{1/E_j\}_{j \text{ even}}$ is an e_n-sequence for the natural shift $(\alpha(m), \beta(m), \gamma(m))$.

In Theorem 8, what is meant by saying $\{1/E_j\}_{j \text{ even}}$ is an e_n-sequence? For that matter, what is meant by saying $\{u_j\}_{j \text{ of the form } F}$ is an e_n-sequence? This simply means:

(a) $f(n, j)$ is of the form F for $0 \leq j \leq n$,

(b) $\{u_{f(n,j)}\}_{j=0}^{n}$ are all m-integers, and

(c) $\sum_{j=0}^{n} (-1)^j \binom{n}{j} u_{f(n,j)} \equiv 0 \pmod{m^n}$ where $m = \text{LCM}[m_1, m_2, \ldots, m_t] > 1$ with $m_1, m_2, \ldots, m_t$ natural numbers.

Since

$$\left\{\frac{B_{j+1}}{j+1} \cdot \frac{j+1}{B_{j+1}}\right\}_{j \text{ odd}}$$

is an e_n-sequence with shift $(r\alpha_1(m), r\beta_1(m), \gamma(m))$, $r \geq 0$ and $\gamma(m) \geq 0$ for the odd natural number $m = \text{LCM}[m_1, m_2, \ldots, m_t] > 1$, Theorem 7 gives conditions for $\{B_{j+1}/(j+1)\}_{j \text{ odd}}$ to be an e_n-sequence, and $[f(n,j) + 1]/(B_{f(n,j)+1})$ will be an m-integer when

$$\text{GCD}\left(m, \frac{B_{f(n,j)+1}}{f(n,j)+1}\right) = 1.$$

This implies

Theorem 9: Let $m = \text{LCM}[m_1, m_2, \ldots, m_t] > 1$ with $m_1, m_2, \ldots, m_t$ all natural numbers and m odd. If

(a) $f(n, j) = (n - j)r\alpha_1(m) + r\beta_1(m) + \gamma(m)$ is an odd natural number for $0 \leq j \leq n$; and

(b) $r \geq 0$ and $\gamma(m) \geq 0$; and

(c) $\text{GCD}(m, 2^{ir\alpha_1(m)+\gamma(m)+1} - 1) = 1$ or, equivalently,
$\text{GCD}(m, 2^{ir\beta_1(m)+\gamma(m)+1} - 1) = 1$ for $i = 1, 2, \ldots, n$; and

(d) $\left(m, \frac{B_{f(n,j)+1}}{f(n,j)+1}\right) = 1$ for at least one $j = 0, 1, 2, \ldots, n$, then $\left\{\dfrac{f(n, j) + 1}{B_{f(n,j)+1}}\right\}_{j=0}^{n}$ are all m-integers and $\left\{\dfrac{j + 1}{B_{j+1}}\right\}_{j=0}^{\infty}$ is an e_n-sequence with natural shift $(\alpha(m), \beta(m), \gamma(m))$.

5. The Tangent Numbers

The *tangent numbers* $\{T_j\}_{j=0}^{\infty}$ are defined by the generating function

$$\tan x = \sum_{j=0}^{\infty} \frac{T_j x^j}{j!}. \tag{33}$$

It is well known that $T_{2j} = 0$, $j \geq 0$, and

$$T_{2n-1} = (-1)^{n-1}4^n(4^n - 1)\frac{B_{2n}}{2n} \text{ is a positive integer.} \tag{34}$$

For a discussion of these numbers, see [12], page 273. Theorem 4 together with these observations gives

Theorem 10: Let $m = \text{LCM}[m_1, m_2, \ldots, m_t] > 1$ with $m_1, m_2, \ldots, m_t$ natural numbers be an odd number and suppose

$$f(n, j) = (n - j)r\alpha_1(m) + r\beta_1(m)j + \gamma(m) \geq 1$$

for $0 \leq j \leq n$, $r \geq 0$, and $\gamma(m) \geq 0$. Then $\{(-1)^{(j-1)/2}T_j\}_{j \text{ odd}}$ is an e_n-sequence with the natural shift $(r\alpha_1(m), r\beta_1(m), \gamma(m))$.

6. Miscellaneous Results

A formula analogous to (12) for Bernoulli polynomials is

$$\sum_{i=1}^{N} i^n = \frac{1}{n + 1}(B_{n+1}(N + 1) - B_{n+1}), \tag{35}$$

where both n and N are natural numbers (see [16], page 26). Let

$$f(n, j) = f_j = (n - j)r\alpha_1(m) + r\beta_1(m) + \gamma(m),$$

where $m = \text{LCM}[m_1, m_2, \ldots, m_t] > 1$ and $m_1, m_2, \ldots, m_t$ are natural numbers. In (35), replace n by f_j (so that $f_j \geq 0$) and to this apply the operator

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j}$$

so that

$$\sum_{i=1}^{N} \sum_{j=0}^{n} (-1)^i \binom{n}{j} i^{f_j} = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \left[\frac{B_{f_j+1}(N + 1) - B_{f_j+1}}{f_j + 1}\right]. \tag{36}$$

Using Theorem 1, this implies

Theorem 11: Let $m = \text{LCM}[m_1, m_2, \ldots, m_t] > 1$ with $m_1, m_2, \ldots, m_t$ natural numbers, and let

$$f_j = f(n, j) = (n - j)r\alpha_1(m) + r\beta_1(m) + \gamma(m) \geq 1$$

for $0 \leq j \leq n$, $r \geq 0$, and $\gamma(m) \geq 0$. If x is an m-integer, then

$$\left\{\frac{B_{f_j+1}(x) - B_{f_j+1}}{f_j + 1}\right\}_{j=0}^{\infty}$$

are all m-integers and

$$\left\{\frac{B_{j+1}(x) - B_{j+1}}{j + 1}\right\}_{j=0}^{\infty}$$

is an e_n-sequence with the natural shift $(\alpha(m), \beta(m), \gamma(m))$. Here, n is a natural number.

Now $B_{2k+1} = 0$ for $k = 1, 2, 3, \ldots$, so that, if $f(n, j) + 1 \geq 3$ is an odd number, then

$$\left\{\frac{B_{f(n,j)+1}}{f(n, j) + 1}\right\}_{j=0}^{n} \text{ are all } m\text{-integers and } \left\{\frac{B_{j+1}(x)}{j + 1}\right\}_{j=0}^{n} \text{ is an } e_n\text{-sequence.}$$

With these observations, Theorems 11 and 7 yield

Theorem 12: Let $m = \mathrm{LCM}[m_1, m_2, \ldots, m_t] > 1$ with $m_1, m_2, \ldots, m_t$ natural numbers, and suppose

$$f(n, j) = (n - j)r\alpha_1(m) + r\beta_1(m) + \gamma(m) \geq 1$$

for $0 \leq j \leq n$, $r \geq 0$, and $\gamma(m) \geq 0$. Suppose also that x is an m-integer. If $f(n, j) + 1 \geq 3$ and $f(n, j)$ is even for $0 \leq j \leq n$, or if

$$\mathrm{GCD}\left(m, 2^{ir\alpha_1(m) + \gamma(m) + 1} - 1\right) = 1 \text{ or, equivalently, } \mathrm{GCD}\left(m, 2^{ir\beta_1(m) + \gamma(m) + 1} - 1\right) = 1$$

for $1 \leq i \leq n$, then

$$\left\{\frac{B_{f(n,j)+1}(x)}{f(n, j) + 1}\right\}_{j=0}^{\infty} \text{ are all } m\text{-integers}$$

and

$$\left\{\frac{B_{j+1}(x)}{j + 1}\right\}_{j=0}^{\infty}$$

is an e -sequence with natural shift $(r\alpha_1(m), r\beta_1(m), \gamma(m))$. Here, n is a natural number.

Varieties using these results can easily be made. For example, in Theorem 12, since x is an m-integer, $-x$ is also an m-integer, and it follows that

$$\left\{\frac{B_{j+1}(x) - B_{j+1}(-x)}{j + 1}\right\}_{j=0}^{\infty}$$

is an e_n-sequence. Here, the even powers of x are missing since

$$\frac{B_{j+1}(x) - B_{j+1}(-x)}{j + 1}$$

is an odd function of x. By the same reasoning

$$\left\{\frac{B_{j+1}(x) + B_{j+1}(-x)}{j + 1}\right\}_{j=0}^{\infty}$$

is an e_n-sequence. Here, the odd powers of x are missing since

$$\frac{B_{j+1}(x) + B_{j+1}(-x)}{j + 1}$$

is an even function of x. Similar remarks can, of course, be made concerning the Euler polynomials.

7. Binomial Rings

As has been seen, the Product Theorem allows for various combinations involving e-sequences. This will now be investigated.

Definition 5: A sequence $\{w_j\}_{j=0}^{\infty}$ is said to be *well behaved to* k where k is a natural number with respect to $m > 1$ and α and β integers provided for every natural number $n \leq k$ it is an e_{n-i}-sequence with shift $(\alpha, \beta, \beta i + \gamma)$ for $i = 0, 1, 2, \ldots, n - 1$ and it is an e_i-sequence with shift $(\alpha, \beta, (n - i)\alpha + \gamma)$

for $i = 1, 2, \ldots, n$ where the conditions to be a shift are satisfied in each instance and γ is arbitrary. This means that γ is chosen from the set of all integers S which is such that if $\gamma_0 \in S$, $\beta i + \gamma_0 \in S$ for $i = 0, 1, 2, \ldots, n - 1$ and $(n - i)\alpha + \gamma_0 \in S$ for $i = 1, 2, \ldots, n$ and the shift conditions are satisfied for all values $\gamma \in S$ for the given values α and β.

Note that if $\{w_j\}_{j=0}^{\infty}$ is a well-behaved sequence to k and if $k_1 < k$ is any natural number, then $\{w_j\}_{j=0}^{\infty}$ is also well behaved to k_1. When the phrase "$\{w_j\}_{j=0}^{\infty}$ is a well-behaved sequence" is used, it will be supposed "to arbitrary k a natural number." Unless otherwise stated, the shift that will be used for well-behaved sequences is $(r\alpha_1(m), r\beta_1(m), \gamma(m))$ where r and $\gamma(m)$ are whole numbers.

One of the examples of a well-behaved sequence for any k a natural number that has been given is the sequence $\{E_j\}_{j=0}^{\infty}$ of Euler numbers with the shift $(r\alpha_1(m), r\beta_1(m), \gamma(m))$ for r a fixed whole number and γ an arbitrary whole number with $m = \text{LCM}[m_1, m_2, \ldots, m_t] > 1$ with $m_1, m_2, \ldots, m_t$ natural numbers.

It is clear by the Product Theorem that the "product"

$$(\{u_j\}_{j=0}^{\infty}\{v_j\}_{j=0}^{\infty} = \{u_j v_j\}_{j=0}^{\infty})$$

of well-behaved sequences all with respect to m, $\alpha(m)$ and $\beta(m)$ is also a well-behaved sequence. Indeed, it is this that motivated Definition 5.

Definition 6: Let k, $m = \text{LCM}[m_1, m_2, \ldots, m_t] > 1$ and $m_1, m_2, \ldots, m_t$ be natural numbers. Let

$$R\binom{k}{m} = \{(x_0, x_1, \ldots, x_k) \,|\, x_0, x_1, \ldots, x_k \text{ are all } m\text{-integers}\}$$

and suppose

$$(x_0, x_1, \ldots, x_k),\ (y_0, y_1, \ldots, y_k) \in R\binom{k}{m}.$$

Then

(a) $(x_0, x_1, \ldots, x_k) = (y_0, y_1, \ldots, y_k)$ provided $x_i \equiv y_i \pmod{m^k}$ for $0 \le i \le k$;

(b) $(x_0, x_1, \ldots, x_k) + (y_0, y_1, \ldots, y_k) = (x_0 + y_0, x_1 + y_1, \ldots, x_k + y_k)$;

(c) $(x_0, x_1, \ldots, x_k) \cdot (y_0, y_1, \ldots, y_k) = (x_0y_0, x_1y_1, \ldots, x_ky_k)$;

(d) If α is any m-integer, $\alpha(x_0, x_1, \ldots, x_k) = (\alpha x_0, \alpha x_1, \ldots, \alpha x_k)$;

(e) Let n be any integer. If $x_1^n, x_2^n, \ldots, x_k^n$ all exist $\pmod{m^k}$, then $(x_0, x_1, \ldots, x_k)^n = (x_0^n, x_1^n, \ldots, x_k^n)$.

It is clear that $R\binom{k}{n}$ is a commutative ring with identity $e = (1, 1, \ldots, 1)$. $R\binom{k}{m}$ is called the ring of $(k + 1)$-*tuples of m-integers* $\pmod{m^k}$ and, furthermore, by the Product Theorem, there exist subrings $B\binom{k}{m}$ of $R\binom{k}{m}$ such that if $(x_0, x_1, \ldots, x_k) \in B\binom{k}{m}$ then

$$\sum_{j=0}^{k} (-1)^i \binom{k}{j} x_j \equiv 0 \pmod{m^k}. \tag{37}$$

Any such subring of $R\binom{k}{m}$ is called a *binomial ring*.

Let $\{w_j\}_{j=0}^{\infty}$ be a well-behaved sequence. It is clear that

$$(w_{f(k,0)}, w_{f(k,1)}, \ldots, w_{f(k,k)})$$

generates a binomial ring. These observations establish

Theorem 13: Let $\{x_{ij}\}_{j=0}^{\infty}$ for $1 \le i \le t$ all be well-behaved sequences to k with respect to $m = \text{LCM}[m_1, m_2, \ldots, m_t] > 1$ and fixed $\alpha(m)$ and $\beta(m)$. Let $g(x_1, x_2, \ldots, x_t)$ be a polynomial with m-integer coefficients. Let $y_{ij} = x_{if(k,j)}$. Then

$(g(y_{10}, y_{20}, \ldots, y_{t0}), g(y_{11}, y_{21}, \ldots, y_{t1}), \ldots, g(y_{1k}, y_{2k}, \ldots, y_{tk}))$ is an element of a binomial ring.

Definition 7: An element $(x_0, x_1, \ldots, x_k) \in R\binom{k}{m}$ is said to be *principal* provided $(x_0 x_1 \ldots x_k, m) = 1$.

It is clear that if $x = (x_0, x_1, \ldots, x_k)$ is a principal element of $R\binom{k}{m}$, then $\{x, x^2, x^3, \ldots\}$ is a cyclic group under multiplication. Furthermore, it is the principal elements that have multiplicative inverses.

Suppose that $\{w_j\}_{j=0}^{\infty}$ is a well-behaved sequence to k with respect to $m = \text{LCM}[m_1, m_2, \ldots, m_t] > 1$, $\alpha(m)$, and $\beta(m)$. Suppose also that $\{a_i\}$, $\{b_i\}$, and $\{i_q\}$ are all sequences of whole numbers. Then $\{w_{j+b_{i_q}}\}_{j=0}^{\infty}$ is a well-behaved sequence to k. Let α_i, β_i, c_i, d, and g_i be any m-integers. It follows that

$$\left\{\left(\left(\sum_i \prod_q \left(a_{i_q}^j b_{i_q} w_{j+b_{i_q}}^{a_{iq}} + g_i\right)\right) + d + fc_i\right)^n\right\}_{j=0}^{\infty} \tag{38}$$

is well behaved to k with respect to m, α, and β. Here, the sum and the product are finite and $f \equiv 0 \pmod{m^k}$. Other variations besides (38) can, of course, be given.

As has been seen, $\{E_j\}_{j=0}^{\infty}$ is well behaved to any k a natural number for $m > 1$ an odd number with shift $(r\alpha_1(m), r\beta_1(m), \gamma(m))$ for r and $\gamma(m)$ whole numbers.

As an example of a binomial ring constructed from the Euler numbers, let $m = 5 = \text{LCM}[5]$ and $k = 3$. Here, using the natural shift

$$\begin{aligned} f(3, j) &= (3 - j)r(\phi(5) + h(5)]^g + r[h(5)]^g j + \gamma(5) \\ &= (3 - j)5 + j + 1 = 16 - 4j, \end{aligned}$$

where $r = g = \gamma = 1$. Here, γ was chosen to be 1 since, for even γ, the corresponding Euler number is 0, and this is trivial. Other choices can, of course, be made for r, g, and $\gamma(m)$. For the above choices,

$$\begin{aligned} E_{16} &= 1\ 9\ 3\ 9\ 1\ 5\ 1\ 2\ 1\ 4\ 5 \equiv 20 \pmod{5^3}, \\ E_{12} &= 2\ 7\ 0\ 2\ 7\ 6\ 5 \equiv 15 \pmod{5^3}, \\ E_8 &= 1\ 3\ 8\ 5 \equiv 10 \pmod{5^3}, \\ E_4 &= 5 \equiv 5 \pmod{5^3}. \end{aligned}$$

Thus,

$$(20, 15, 10, 5) \text{ is a member of a binomial ring } B\binom{3}{5}.$$

Since (x, x, x, x) is also a member, it follows that

$$(20 + x)^n - 3(15 + x)^n + 3(10 + x)^n - (5 + x)^n \equiv 0 \pmod{125}$$

for n any whole number and x any integer.

To construct another element of such a $B\binom{3}{5}$, let $r = g = 1$ and $\gamma = 3$. Then $f(3, j) = 18 - 4j$, so that

$$\begin{aligned} E_{18} &= -2\ 4\ 0\ 4\ 8\ 7\ 9\ 6\ 7\ 5\ 4\ 4\ 1 \equiv 59 \pmod{125}, \\ E_{14} &= -1\ 9\ 9\ 3\ 6\ 0\ 9\ 8\ 1 \equiv 19 \pmod{125}, \\ E_{10} &= -5\ 0\ 5\ 2\ 1 \equiv 104 \pmod{125}, \\ E_6 &= -6\ 1 \equiv 64 \pmod{125}. \end{aligned}$$

Thus,

$$(59, 19, 104, 64) \text{ is a member of a } B\binom{3}{5}.$$

Combining this with the previous element, for x and y any integers, m and n any whole numbers,

$$(20 + x)^m(59 + y)^n - 3(15 + x)^m(19 + y)^n + 3(10 + x)^m(104 + y)^n - (5 + x)^m(64 + y)^n \equiv 0 \pmod{125}.$$

This can actually be made a little stronger. If

$$(20 + x,\ 15 + x,\ 10 + x,\ 5 + x) \text{ and } (59 + y,\ 19 + y,\ 104 + y,\ 64 + y)$$

are both principal, then m and n can be any integers.

8. Some Additional Results with $(\mathrm{mod}\{\mathrm{LCM}[m_1, m_2, \ldots, m_t]\}^n)$

The examples in this paper have been concerned with congruences $(\mathrm{mod}\ m^n)$. The case, with $m = p = \mathrm{LCM}[p]$ and p a prime number, is, of course, well known in connection with Kummer's congruences. Some additional examples will be given here.

The natural shift $(\alpha(m), \beta(m), \gamma(m))$ with $m = \mathrm{LCM}[m_1, m_2, \ldots, m_t]$ will be used. Here

(39) $$\alpha(m) = r\alpha_1(m),\ \beta(m) = r\beta_1(m),$$

with

(40) $$\alpha_1(m) = r_1[\phi(m_1) + h(m_t)]^{g_1} + s_1 = r_2[\phi(m_2) + h(m_2)]^{g_2} + s_2 = \cdots = r_t[\phi(m_t) + h(m_t)]^{g_t} + s_t$$

and

(41) $$\beta_1(m) = r_1[h(m_1)]^{g_1} + s_1 = r_2[h(m_2)]^{g_2} + s_2 = \cdots = r_t[h(m_t)]^{g_t} + s_t,$$

for some integers $r_1, r_2, \ldots, r_t, s_1, s_2, \ldots, s_t$, and some natural numbers $g_1, g_2, \ldots, g_t$. As was remarked earlier, special care is needed for any of the r's or s's to be negative. It will be supposed that $\alpha_1(m)$, $\beta_1(m)$, $r \neq 0$ and $\alpha_1(m) \neq \beta_1(m)$ to keep the results from being trivial.

First, an example using Theorem 3 will be given. Let $m = 15 = \mathrm{LCM}[3, 5]$ and $n = 3$. In this case, $\phi(3) = 2$ and $\phi(5) = 4$ so that $r_1, r_2; s_1, s_2; g_1, g_2$ are required such that

(42) $$r_1[2 + 1]^{g_1} + s_1 = r_2[4 + 1]^{g_2} + s_2 \text{ and } r_1 \cdot 1^{g_1} + s_1 = r_2 \cdot 1^{g_2} + s_2.$$

Clearly, a choice is $r_1 = 2$, $r_2 = 1$; $s_1 = 0$, $s_2 = 1$; $g_1 = g_2 = 1$ and $r = 1$ so that $\alpha(m) = 6$ and $\beta(m) = 2$ so that $f(3, j) = (3 - j) \cdot 6 + 2j + \gamma = 18 - 4j + \gamma$ so that Theorem 3 gives

(43) $$\sum_{j=0}^{3} (-1)^i \binom{3}{j} E_{18-4j+\gamma} \equiv 0 \pmod{15^3} \text{ where } \gamma \text{ is a whole number.}$$

Evidently, other choices for $r_1, r_2, s_1, s_2, g_1, g_2$ in (41) can be made.

On the other hand, if $m = 15 = \mathrm{LCM}[15]$, then

$$f(3, j) = (3 - j)r[\phi(15) + h(15)]^g + r[h(15)]^g_j + \gamma.$$

Let $r = g = 1$ so that

$$f(3, j) = (3 - j)(9) + j + \gamma = 27 - 8j + \gamma$$

and

(44) $$\sum_{j=0}^{3} (-1)^i \binom{3}{j} E_{27-8j+\gamma} \equiv 0 \pmod{15^3}.$$

An example using Theorem 7 is given by $m = 35 = \mathrm{LCM}[5, 7]$. Here, $\phi(5) = 4$, $h(5) = 1$, $\phi(7) = 6$, and $h(7) = 1$ so that r_1, r_2, s_1, and s_2 are needed such that

(45) $$5r_1 + s_1 = 7r_2 + s_2,$$ $$r_1 + s_1 = r_2 + s_2. \quad (\text{Here, } g_1 = g_2 = 1.)$$

A choice for these numbers is $r_1 = 3$, $r_2 = 2$, $s_1 = 0$, and $s_2 = 1$. This gives $\alpha_1(m) = 15$ and $\beta_1(m) = 3$. Choose $r = 1$. From Theorem 7, it is required that $(35, 2^{3+\gamma+1} - 1) = 1$ and $(1 - j)15 + 3j + \gamma + 1$ is even. In this case, $n = 1$. A choice for $\gamma = \gamma(m)$ satisfying this is $\gamma = 6$. Thus, Theorem 7 says that

$$\left\{\frac{B_{22-12j}}{22 - 12j}\right\}_{j=0}^{1} \text{ are both 35-integers and } \sum_{j=0}^{1}(-1)^j\binom{1}{j}\frac{B_{22-12j}}{22 - 12j} \equiv 0 \pmod{35}.$$

Notice that another congruence (mod 35) can easily be given by letting $m = 35 = \mathrm{LCM}[35]$. In this case, $\phi(35) = 24$ and $h(35) = 1$. Thus, a choice of $\alpha(35) = 25$ and $\beta(35) = 1$. To satisfy the hypothesis of Theorem 7, it is required that $(35, 2^{1+\gamma+1} - 1) = 1$ and $(1 - j) \cdot 25 + j + \gamma + 1 = 26 - 24j +$ be even. $\gamma = 0$ works. Thus, according to Theorem 7,

$$\left\{\frac{B_{26-24j}}{26 - 24j}\right\}_{j=0}^{1} \text{ are 35-integers and } \sum_{j=0}^{1}(-1)^j\binom{1}{j}\frac{B_{26-24j}}{26 - 24j} \equiv 0 \pmod{35}.$$

More generally, let a and b be natural numbers such that $m = \mathrm{LCM}[a, b] > 1$ is odd. Then it is required to find r_1, r_2, s_1, s_2, for $g_1 = g_2 = 1$ such that

$$\begin{aligned} &r_1[\phi(a) + h(a)] + s_1 = r_2[\phi(b) + h(b)] + s_2, \\ &r_1 h(a) + s_1 = r_2 h(b) + s_2. \end{aligned} \tag{46}$$

A choise for r_1 and s_1 satisfying this is

$$r_1 = \frac{\mathrm{LCM}[\phi(a),\ \phi(b)]}{\phi(a)} \quad \text{and } s_1 = 0.$$

For this choice,

$$\alpha(m) = \frac{r\ \mathrm{LCM}[\phi(a),\ \phi(b)][\phi(a) + h(a)]}{\phi(a)}$$

and

$$\beta(m) = \frac{r\ \mathrm{LCM}[\phi(a),\ \phi(b)]h(a)}{\phi(a)},$$

so that, by Theorem 7, if

$$\left(\mathrm{LCM}[a,\ b],\ 2^{\frac{ir\ \mathrm{LCM}[\phi(a),\ \phi(b)]\ h(a)}{\phi(a)} + \gamma + 1} - 1\right) = 1 \quad \text{for } i = 1, 2, 3, \ldots, n,$$

then

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{B_{\left\{r\,\mathrm{LCM}[\phi(a),\,\phi(b)]j + \frac{nr\ \mathrm{LCM}[\phi(a),\ \phi(b)]\ h(a)}{\phi(a)} + \gamma + 1\right\}}}{r\ \mathrm{LCM}[\phi(a),\ \phi(b)]j + \dfrac{nr\ \mathrm{LCM}[\phi(a),\ \phi(b)]h(a)}{\phi(a)} + \gamma + 1} \tag{48}$$

$$\equiv 0 \pmod{\{\mathrm{LCM}[a,\ b]\}^n}. \quad [\text{Here, } \gamma = j(m)].$$

Notice that since there exist a and b such that

$$\mathrm{LCM}[\phi(a),\ \phi(b)] \neq \phi(\mathrm{LCM}[a,\ b])$$

(for example, $a = 15$ and $b = 35$) it follows that (48) is essentially different from what would be obtained simply by letting $m = \mathrm{LCM}[m]$ for $m = \mathrm{LCM}[a, b]$.

The reader might enjoy examining the congruences obtained from

$$\begin{aligned} m = 105 &= \mathrm{LCM}[105] = \mathrm{LCM}[3,\ 5,\ 7] = \mathrm{LCM}[15,\ 7] = \mathrm{LCM}[21,\ 5] \\ &= \mathrm{LCM}[3,\ 35] = \mathrm{LCM}[15,\ 35] = \mathrm{LCM}[21,\ 35] = \mathrm{LCM}[15,\ 21] \end{aligned}$$

for these various LCM-partitions of 105.

Acknowledgment

The author would like to thank an anonymous referee for numerous helpful suggestions.

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AMS Classification numbers: 11B68, 11B48, 11B80

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Our home during the Conference was a University dormitory. John Burnet Hall, formerly a hotel, and still providing the comfort of such. Colleges and Universities have the reputation of offering dull, institutionalized fare. Our food, taken at the dorm's cafeteria, constituted an enjoyable counterexample.

St Andrews is an ancient institution. And during its nearly six centuries of existence, it has maintained vigorous scholarly impact across the whole academic spectrum. St. Andrews has been called "a gem of a University"—uniquely Scottish by history and beautiful location, yet unusually cosmopolitan.

The Conference's social events rounded off, and enhanced, our academic sessions. The traditional mid-conference's afternoon excursion took us to Falkland, a Renaissance Palace, which grew out of the medieval Falkland Castle. At once did we get lured into the quaintness of an historically rich palace and became enchanted by the charming multi-coloredness of the garden.

To convey the congenial and happy atmosphere at our Conference—dinner adequately would require a vocabulary far richer than mine. Interspersed with inspirational short talks and remarks, animated by delicious banquet fare and, most of all, by having our whole group gathered together, it was simply delightful.

And, finally, the Conference itself.

Erudite and always carefully prepared papers ranged over the heights and depths of "purity" and "applicability," once more illustrating the startling way in which these two facets of mathematics are duals of each other. And while we speak with many different accents, we understand each other on a much more significant level. Almost immediately, friendships blossomed or ripened, as the love of our discipline and the enthusiasm for it were written over all the faces of the "Fibonaccians" as some of us like to refer to ourselves. That one week in Scotland, kindled by the serenity of the Scottish landscape and enhanced by the spirit of our Scottish hosts and co-mathematicians, gave us experiences which were both mentally enriching and personally heartwarming.

Finally, it was "farewell." But it is with much happiness that we can say: "*Auf Wiedersehen* in two years at Pullman, Washington."

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