MULTINOMIAL AND Q-BINOMIAL COEFFICIENTS MODULO 4 AND MODULO P

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1. INTRODUCTION

Hexel and Sachs [3] examined the n^{th} row of Pascal's triangle and worked out formulas for the number of occurrences of each residue modulo p, where p is any prime. For $p > 3$ the formulas are very involved. Davis & Webb [1] recently considered the same problem modulo 4, and they pointed out that a composite modulus requires an approach different from the one in [3]. To date, 4 is the only composite modulus for which formulas for the number of occurrences of each residue have been obtained. It appears to be very difficult to find results of this type for arbitrary composite moduli.

The purpose of the present paper is to extend the results of [1] and [3] to multinomial and q-binomial coefficients. Thus, in section 3 we examine the q-binomial coefficients $\left[\begin{smallmatrix} n \\ r \end{smallmatrix}\right]$, $0 \le r \le n$, and determine the number of occurrences of each residue modulo 4. In section 4 we consider the same problem modulo p, and we obtain explicit formulas for $p = 3$. For $p \ge 3$, we show how formulas can be worked out in terms of the results of [3]. Similarly, in section 6 we examine the multinomial coefficients $(n_1, n_2, \ldots, n_r)$ such that $n_1 + n_2 + \cdots + n_r = n$, and we find formulas that enable us to compute the number of occurrences of each residue modulo 4. In section 7 we consider the same problem modulo p. An explicit formula for $p = 3$ is determined, and formulas for $p \ge 3$ are found in terms of the results of [3]. In sections 2 and 5 we state the basic properties of the q-binomial and multinomial coefficients that we need, and we also explain the notation used in this paper.

2. Q-BINOMIAL COEFFICIENTS: PRELIMINARIES

The q-binomial coefficient is defined by

$$\text{(2.1)} \qquad \begin{bmatrix} n \\ r \end{bmatrix} = \prod_{j=1}^{r} \frac{q^{n-j+1} - 1}{q^j - 1}$$

for q an indeterminate and n a nonnegative integer. When considering $\left[\begin{smallmatrix} n \\ r \end{smallmatrix}\right]$ modulo j, for any j, unless otherwise stated q will always be a rational number, $q = u/v$, with $\gcd(u,v) = \gcd(u,j) = \gcd(v,j) = 1$ The q-binomial coefficient is a polynomial in q, and for $q = 1$ it reduces to the ordinary binomial coefficient. It is clear from (2.1) that, for $r > 0$,

$$\text{(2.2)} \qquad \begin{bmatrix} n \\ r \end{bmatrix} = \begin{bmatrix} n-1 \\ r-1 \end{bmatrix} + q^r \begin{bmatrix} n-1 \\ r \end{bmatrix},$$

$$\text{(2.3)} \qquad \begin{bmatrix} n \\ r \end{bmatrix} = \begin{bmatrix} n \\ n-r \end{bmatrix}.$$

As much as possible, we shall use the notation of [1]. Thus, if

$$(2.4) \qquad n = \sum_{i=0}^{k} a_i 2^i \quad (\text{each } a_i = 0 \text{ or } 1),$$

we define

$$B(n) = \sum_{i=0}^{k} a_i .$$

Similarly, we define

$$C(n) = \sum_{i=0}^{k} c_i, \quad D(n) = \sum_{i=0}^{k} d_i,$$

where $c_i = 1$ if and only if $a_{i+1} = 1$, $a_i = 0$, and $d_i = (a_{i+1})(a_i)$. That is, $C(n)$ is the number of "10" blocks and $D(n)$ is the number of "11" blocks in the base 2 representation of n. The same notation was used in [1].

We shall also use the notation

$$\left\{ \begin{matrix} n \\ r \end{matrix} \right\}_w = j \text{ if and only if } \begin{bmatrix} n \\ r \end{bmatrix} \equiv j \pmod{w} \quad (0 \le j \le w-1),$$

and $N_1^{(w)}(q;n)$ is the number of ones, $N_2^{(w)}(q;n)$ is the number of twos, $N_3^{(w)}(q;n)$ is the number of threes, etc., in the set

$$\left\{ \begin{matrix} n \\ 0 \end{matrix} \right\}_w, \ \left\{ \begin{matrix} n \\ 1 \end{matrix} \right\}_w, \ldots, \left\{ \begin{matrix} n \\ n \end{matrix} \right\}_w .$$

In [2] Fray proved a rule for finding the highest power of a prime p dividing $\left[\begin{smallmatrix} n \\ r \end{smallmatrix} \right]$. The following lemma is a special case of that rule.

Lemma 2.1: Let p be a prime number and let e be the smallest positive integer such that $q^e \equiv 1 \pmod{p}$. Write n, r, and $n - r$ uniquely as

$$(2.5) \qquad n = a_{-1} + e \cdot a = a_{-1} + e \sum_{i=0}^{k} a_i p^i \quad (0 \le a_{-1} < e, \ 0 \le a_i < p),$$

$$(2.6) \qquad r = b_{-1} + e \cdot b = b_{-1} + e \sum_{i=0}^{k} b_i p^i \quad (0 \le b_{-1} < e, \ 0 \le b_i < p),$$

$$(2.7) \qquad n - r = w_{-1} + e \sum_{i=0}^{k} w_i p^i \quad (0 \le w_{-1} < e, \ 0 \le w_i < p).$$

We can write

$$\begin{aligned} b_{-1} + w_{-1} &= e\varepsilon_0 + a_{-1}, \\ \varepsilon_0 + b_0 + w_0 &= p\varepsilon_1 + a_0, \\ &\cdots \\ \varepsilon_{k-1} + b_{k-1} + w_{k-1} &= p\varepsilon_k + a_{k-1}, \\ \varepsilon_k + b_k + w_k &= a_k, \end{aligned}$$

with each $\varepsilon_i = 0$ or 1. Then $\left[\begin{smallmatrix} n \\ r \end{smallmatrix} \right]$ is relatively prime to p if and only if $\varepsilon_i = 0$ for each i.

Note that (2.5)-(2.7) are possible by the division algorithm. Also note that when $p = 2$, we have $e = 1$ and $a_{-1} = b_{-1} = w_{-1} = 0$.

Fray [2] also proved the following useful lemma.

Lemma 2.2: Let n and r have expansions (2.5) and (2.6). Then

$$\begin{bmatrix} n \\ r \end{bmatrix} \equiv \begin{bmatrix} a_{-1} \\ b_{-1} \end{bmatrix} \binom{a}{b} \equiv \begin{bmatrix} a_{-1} \\ b_{-1} \end{bmatrix} \binom{a_o}{b_0} \binom{a_1}{b_1} \cdots \binom{a_k}{b_k} \pmod{p}.$$

The second congruence of Lemma 2.2 follows from a well-known theorem of Lucas.

Let $\alpha_j(p;n)$ denote the number of q-binomial coefficients $\left[\begin{smallmatrix} n \\ r \end{smallmatrix}\right]$, $r = 0, 1, \ldots, n$, divisible by exactly p^j (that is, divisible by p^j but not by p^{j+1}). Fray [2] proved that if n has expansion (2.5), then

$$\alpha_0(p;n) = (a_{-1}+1)(a_0+1)\cdots(a_k+1).$$

In particular, for $p = 2$, let

$$\alpha_j(n) = \alpha_j(2;n),$$

so

$$\alpha_0(n) = 2^{B(n)}.$$

The writer [4] proved that

$$\alpha_1(n) = \begin{cases} C\left(\dfrac{n-a_0}{2}\right)2^{B(n)-1} & \text{if } q \equiv 3 \pmod{4}, \\ C(n)2^{B(n)-1} & \text{if } q \equiv 1 \pmod{4}. \end{cases} \tag{2.8}$$

3. Q-BINOMIAL COEFFICIENTS MODULO 4

We shall use the notation of section 2, and for convenience we let

$$\begin{Bmatrix} n \\ r \end{Bmatrix} = \begin{Bmatrix} n \\ r \end{Bmatrix}_4,$$

$$N_j(n) = N_4^{(j)}(q;n).$$

Also define

$$N(n) = \left(N_1(n), N_2(n), N_3(n)\right).$$

First we take care of some trivial cases. If $q \equiv 0$ (mod 4), we see from (2.1) that

$$\begin{Bmatrix} n \\ r \end{Bmatrix} = 1 \quad (r = 0, 1, \ldots, n)$$

If $q \equiv 2$ (mod 4), we see from (2.1) that

$$\begin{Bmatrix} n \\ 0 \end{Bmatrix} = \begin{Bmatrix} n \\ n \end{Bmatrix} = 1, \quad \begin{Bmatrix} n \\ r \end{Bmatrix} = 3 \quad (r = 1, \ldots, n-1)$$

If $q \equiv 1 \pmod 4$, then

$$\left\{ {n \atop r} \right\} \equiv \binom{n}{r} \pmod 4,$$

and the results of [1] can be used.

In the remainder of this section we shall assume $q \equiv 3 \pmod 4$. We shall also use the notation of section 2.

We know from (2.3) that

$$N_2(n) = C\left(\frac{n-a_0}{2}\right) 2^{B(n)-1}.$$

Note that

$$C\left(\frac{n-a_0}{2}\right) = \begin{cases} C(n) & \text{if } n \not\equiv 2 \pmod 4, \\ C(n)-1 & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

It is clear from (2.2) that

$$\left\{ {n \atop r} \right\} \equiv \left\{ {n-1 \atop r-1} \right\} + (-1)^r \left\{ {n-1 \atop r} \right\} \pmod 4, \tag{3.1}$$

and the following lemmas are clear from (2.1), (2.5), and Lemma 2.1.

Lemma 3.1: When $k > 1$, $N(2^k) = (2, 1, 0)$.

Lemma 3.2: Let $n = 2^k + L$, where $0 < L < 2^k$. Then

$$\left\{ {n \atop r} \right\} \equiv 0 \pmod 2 \quad (L < r < 2^k).$$

The analogous results for ordinary binomial coefficients are proved in [1]. By (3.1) and Lemmas 3.1 and 3.2, we see that the q-binomial Pascal triangle modulo 4 for $q \equiv 3 \pmod 4$ has the following form:

$$\begin{array}{lc} & 1 \\ & 1\ 1 \\ & 1\ 0\ 1 \\ & 1\ 1\ 1\ 1 \\ & 1\ 0\ 2\ 0\ 1 \\ & \cdots \\ (2^k \text{ row}) & 1\ 0 \ldots 0\ 2\ 0 \ldots 0\ 1 \\ (2^k+1 \text{ row}) & 1\ 1\ 0 \ldots 0\ 2\ 2\ 0 \ldots 0\ 1\ 1 \end{array}$$

By using (3.1) and comparing this triangle with Pascal's triangle modulo 4 (see [1]), we see that the two triangles satisfy the same recursive relations. That is, in Part 1 and Part 2 of [1], we can replace $\langle \cdots \rangle$ by $\{\cdots\}$. We shall not reproduce all those relations here, but we note the following.

Lemma 3.3: Suppose $n = 2^k + L,\ 0 \le L < 2^k$

(a) If $L < 2^{k-1}$, then

$$\left\{ {n \atop r} \right\} \text{ is } \begin{cases} = \left\{ {L \atop r} \right\} & \text{if } 0 \le r \le L, \\ \equiv 0 \pmod 2 & \text{if } L < r < 2^k. \end{cases}$$

(b) If $2^{k-1} \le L < 2^k$, then

$$\left\{ {n \atop r} \right\} \text{ is } \begin{cases} = \left\{ {L \atop r} \right\} & \text{if } 0 \le r < 2^{k-1}, \\ \equiv \left\{ {L \atop r} \right\} + 2\left\{ {L \atop r - 2^{k-1}} \right\} \pmod 4 & \text{if } 2^{k-1} \le r \le L, \\ \equiv 0 \pmod 2 & \text{if } L < r < 2^k. \end{cases}$$

Because of the symmetry of the triangle, i.e., property (2.3), we now have all the information we need.

Recall that $D(n) > 0$ if and only if the base 2 representation of n has a "11" block.

Theorem 3.1 If $D(n) = 0$, or $n = 3 + 8m$ with $D(m) = 0$, then $N_1(n) = 2^{B(n)}$ and $N_3(n) = 0$.

Proof: We use induction on n. The theorem is true for $n \le 3$; assume it is true for all nonnegative integers less than n. If n satisfies the hypotheses of the theorem, then $n = 2^k + L$ with $L < 2^{k-1}$, and either $D(L) = 0$ or $L = 3 + 8y$ with $D(y) = 0$. Thus,

$$N_1(L) = 2^{B(L)} \text{ and } N_3(L) = 0.$$

Note that $B(n) = B(L) + 1$. We know

$$\left\{ {n \atop r} \right\} \text{ is } \begin{cases} = \left\{ {L \atop r} \right\} & \text{if } 0 \le r \le L, \\ \equiv 0 \pmod 2 & \text{if } L < r < 2^k. \end{cases}$$

Since $\left\{ {n \atop r} \right\} = \left\{ {n \atop n-r} \right\}$ and $2^k > n/2$, we have

$$N_1(n) = 2N_1(L) = 2^{B(n)} \text{ and } N_3(n) = 0.$$

This completes the proof.

Theorem 3.2: If $D(n) > 0$ and $n \ne 3 + 8m$ with $D(m) = 0$, then $N_1(n) = N_3(n) = 2^{B(n)-1}$.

The proof of Theorem 3.2 is the same as the proof of Theorem 6 in [1], with $\langle \cdots \rangle$ replaced by $\{\cdots\}$. We shall not reproduce it here.

In summary we have:

- If $D(n) = 0$ or $n = 3 + 8m$ with $D(m) = 0$, then

$$N(n) = \left(2^{B(n)},\ C\left(\frac{n - a_0}{2} \right) 2^{B(n)-1},\ 0 \right).$$

- If $D(n) > 0$ and $n \neq 3+8m$ with $D(m) = 0$, then

$$N(n) = \left(2^{B(n)-1},\ C\left(\frac{n-a_0}{2}\right)2^{B(n)-1},\ 2^{B(n)-1}\right).$$

4. Q-BINOMIAL COEFFICIENTS MODULO P

In this section we assume p is an odd prime and n has expansion (2.5). We shall use the following notation:

Let A_j be the number of coefficients a_i $(-1 \le i \le k)$ in (2.5) that are equal to j.

Let t be the order of 2 modulo p, and for m > 0 let $t(m)$ be the smallest nonnegative solution x to $2^x \equiv m \pmod{p}$, if one exists.

Recall that $N_m^{(p)}(q;n)$ is the number of q-binomial coefficients $\left[{n \atop r}\right]$ congruent to m modulo p.

Theorem 4.1 Suppose n has expansion (2.5) with $0 \le a_{-1} \le 1$ and $0 \le a_i \le 2$ for each $i \ge 0$.

(a) If $2^x \equiv m \pmod{p}$ has no solutions x, then $N_m^{(p)}(q;n) = 0$.

(b) If $2^x \equiv m \pmod{p}$ has solutions, then

$$N_m^{(p)}(q;n) = 2^{A_1}\sum_{j=0}^{s}\binom{A_2}{t(m)+jt}2^{A_2-t(m)-jt}, \tag{4.1}$$

where $t(m)+st \le A_2 < t(m)+(s+1)t$.

Proof: We see from Lemma 2.2 that to have $\left[{n \atop r}\right] \equiv m \pmod{p}$ we must have h integers i such that

$$\binom{a_i}{b_i} = \binom{2}{1} \quad \text{and} \quad 2^h \equiv m \pmod{p}.$$

Thus, part (a) is clear. Now $2^h \equiv 2^{t(m)} \pmod{p}$ implies $2h \equiv t(m) \pmod{t}$, so $h = t(m)+jt$ for some j. There are $\binom{A_2}{h}$ ways to have h terms $\binom{2}{1}$; there are two choices for each of the remaining $A_2 - h$ terms $\binom{2}{b_i}$, namely, $b_i = 0$ or $b_i = 2$; there are two choices for each of the A_1 terms $\binom{1}{b_i}$, namely, $b_i = 0$ or $b_i = 1$. Thus, we have (4.1), and the proof is complete.

Corollary: If $p = 3$ [and thus $q \equiv \pm 1 \pmod{3}$], we have

$$N_1^{(3)}(q;n) = \frac{1}{2}\cdot 2^{A_1}(3^{A_2}+1) \quad \text{and} \quad N_2^{(3)}(q;n) = \frac{1}{2}\cdot 2^{A_1}(3^{A_2}-1).$$

Proof: Since $p = 3$, we have all the hypotheses of Theorem 4.1 with $t(1) = 0$, $t(2) = 1$, and $t = 2$. Thus,

$$N_1^{(3)}(q;n) = 2^{A_1}\sum_{j=0}^{s}\binom{A_2}{2j}2^{A_2-2j} = \frac{1}{2}\cdot 2^{A_1}\left[(2+1)^{A_2}+(2-1)^{A_2}\right].$$

The formula for $N_2^{(3)}(q;n)$ is proved in a similar way, thus completing the proof.

By Lemma 2.2, it is clear that

$$N_m^{(p)}(q;n) = \begin{cases} N_m^{(p)}(1;a) & \text{if } a_{-1}=0, \\ 2N_m^{(p)}(1;a) & \text{if } a_{-1}=1, \end{cases}$$

where a is defined by (2.5) and $N_m^{(p)}(1;a)$ is the number of binomial coefficients $\binom{a}{r}$ that are congruent to m modulo p. Thus, when $a_{-1}=0$ or 1, the formulas of [3] can be used to evaluate $N_m^{(p)}(q;n)$. More generally, define $y(r)$ to be the smallest nonnegative solution to

$$\begin{bmatrix} a_{-1} \\ r \end{bmatrix} x \equiv m \pmod{p}.$$

Then the following theorem is clear from Lemma 2.2.

Theorem 4.2: If n has expansion (2.5) and $y(r)$ is defined as above, then

$$N_m^{(p)}(q;n) = \sum_{r=0}^{a_{-1}} N_{y(r)}^{(p)}(1;a).$$

For example, let $p=5$ and $q \equiv 3 \pmod 5$, so $e=4$. We have

$$N_1^{(5)}(3;n) = \begin{cases} N_1^{(5)}(1;a) & \text{if } n \equiv 0 \pmod 4, \\ 2N_1^{(5)}(1;a) & \text{if } n \equiv 1 \pmod 4, \\ 2N_1^{(5)}(1;a) + N_4^{(5)}(1;a) & \text{if } n \equiv 2 \pmod 4, \\ 2N_1^{(5)}(1;a) + 2N_2^{(5)}(1;a) & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Theoretically, then, we can evaluate $N_m^{(p)}(q;n)$ for any q by using Theorem 4.2 and the formulas of [3].

For completeness, we note that for $p=2$ and $q \not\equiv 0 \pmod 2$ we have

$$N_1^{(2)}(q;n) = (a_0+1)(a_1+1)\cdots(a_k+1) = 2^{B(n)},$$

where n has expansion (2.4).

5. MULTINOMIAL COEFFICIENTS: PRELIMINARIES

The multinomial coefficient is defined by

$$(5.1) \qquad (n_1, n_2, \ldots, n_r) = \frac{n!}{n_1!n_2!\cdots n_r!} \qquad (n_1+\cdots+n_r=n).$$

Obviously (5.1) reduces to the ordinary binomial coefficient for $r=2$. In this paper we consider (5.1) for all compositions (ordered partitions) of n into r parts. The order of the terms $n_1, \ldots, n_r$ is important; we are distinguishing between (0, 0, 1, 2) and (1, 0, 0, 2), for example. Note that 0 can be one or more of the parts. It is well known that the number of compositions of n into r parts is $\binom{n+r-1}{n}$.

Fray [2] proved the following rule for determining the highest power of a prime p dividing $(n_1, \ldots, n_r)$.

Lemma 5.1: Let n have base p representation

$$(5.2) \qquad n=\sum_{j=0}^{k} a_j p^j \quad (0 \le a_j < p)$$

and let $n = n_1 + n_2 + \cdots + n_r$. For $i = 1, \ldots, r$, let $n_i = \sum_{j=0}^{k} a_{i,j} p^j \quad (0 \le a_{i,j} < p)$. If

$$\begin{aligned} a_{1,0} + \cdots + a_{r,0} &= p\varepsilon_0 + a_0, \\ \varepsilon_0 + a_{1,1} + \cdots + a_{r,1} &= p\varepsilon_1 + a_1, \\ &\vdots \\ \varepsilon_{k-1} + a_{1,k} + \cdots + a_{r,k} &= a_k, \end{aligned}$$

where each $\varepsilon_i = 0, 1, \ldots,$ or $r-1$. Then the highest power of p dividing $(n_1, n_2, \ldots, n_r)$ is p^s, where $s = \varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_{k-1}$.

We shall use the notation $B(n)$, $C(n)$, and $D(n)$ given in section 2.

Let n have base p expansion (5.2) and let $\theta_j^{(p)}(r; n)$ be the number of multinomial coefficients $(n_1, n_2, \ldots, n_r)$ divisible by exactly p^j. The writer [5] proved

$$\theta_0^{(p)}(r; n) = \binom{a_0 + r - 1}{r-1}\binom{a_1 + r - 1}{r-1} \cdots \binom{a_k + r - 1}{r-1}.$$

For $p = 2$, we have

$$(5.3) \qquad \theta_0^{(2)}(r; n) = r^{B(n)},$$

and the writer [4] proved

$$(5.4) \qquad \theta_1^{(2)}(r; n) = C(n)\binom{r}{2} r^{B(n)-1} + D(n)\binom{r}{3} r^{B(n)-2}.$$

For $w = 0, 1, \ldots, j-1$, we define $N_{r,w}^{(j)}(n)$ as the number of multinomial coefficients $(n_1, n_2, \ldots, n_r)$ such that $(n_1, n_2, \ldots, n_r) \equiv w \pmod{j}$.

6. MULTINOMIAL COEFFICIENTS MODULO 4

The notation for this section comes from sections 2 and 5. For convenience we shall use

$$N_{r,w}(n) = N_{r,w}^{(4)}(n),$$

so that

$$(6.1) \qquad N_{r,2}(n) = \theta_1^{(2)}(r; n)$$

and

$$N_{r,0}(n) = \binom{n+r-1}{n} - N_{r,1}(n) - N_{r,2}(n) - N_{r,3}(n).$$

We also define

$$N_r(n) = \left(N_{r,1}(n),\, N_{r,2}(n),\, N_{r,3}(n)\right).$$

By (5.4) and (6.1), the only problem, then, is to find $N_{r,1}(n)$ and $N_{r,3}(n)$.

Theorem 6.1: If $D(n) = 0$, then $N_{r,1}(n) = r^{B(n)}$ and $N_{r,3}(n) = 0$.

Proof: Suppose $D(n)=0$ and $n=n_1+n_2+\cdots+n_r$. If $(n_1,n_2,\ldots,n_r)\not\equiv 0 \pmod 2$, then by Lemma 5.1 we know if $a_j=0$ then $a_{i,j}=0$ for $i=1,\ldots,r$. Also, if $a_j=1$ then $a_{i,j}=1$ for exactly one i. Since

$$(6.2)\qquad (n_1,n_2,\ldots,n_r)=\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots,$$

and since $D(n-n_1-\cdots-n_j)=0$ for $j=1,\ldots,r$, we see that none of the binomial coefficients on the right side of (6.2) is congruent to 3 modulo 4. Thus, $(n_1,n_2,\ldots,n_r)\equiv 1 \pmod 4$, and the proof is complete.

Hence, if $D(n)=0$, we have

$$N_r(n)=\left(r^{B(n)},\, C(n)\binom{r}{2}r^{B(n)-1},\, 0\right).$$

The situation is much harder if $D(n)>0$. We shall use the following notation:

$$f_j(n,i)=\begin{cases}1 & \text{if } \binom{n}{i}\equiv j \pmod 4,\\ 0 & \text{otherwise.}\end{cases}$$

Since

$$(n_1,n_2,\ldots,n_r)=\binom{n}{n_1}(n_2,\ldots,n_r),$$

we have

$$N_{r,1}(n)=\sum_{i=0}^{n}\left[f_1(n,i)\cdot N_{r-1,1}(n-i)+f_3(n,i)\cdot N_{r-1,3}(n-i)\right].$$

We refine this in the next theorem by using the facts that if $(n_1,n_2,\ldots,n_r)\not\equiv 0 \pmod 2$, then in Lemma 5.1 each $\varepsilon_i=0$, and

$$N_{r,1}(n)+N_{r,3}(n)=r^{B(n)}.$$

Theorem 6.2: Let n have base 2 representation (2.4). Then, for $r\geq 3$,

$$N_{r,1}(n)=N_{r-1,1}(n)+1+\sum_i (r-1)^{B(n-i)}f_3(n,i)+\sum_i\left[f_1(n,i)-f_3(n,i)\right]N_{r-1,1}(n-i),$$

where each sum is over all integers i such that $0<i<n$ and

$$i=\sum_{j=0}^{k}e_j2^j \quad (0\leq e_j\leq a_j)$$

To illustrate Theorem 6.2, suppose

$$(6.3)\qquad n=2^{k-1}+2^k.$$

Then, using Theorem 6.1 and the results of [1], we know $f_3(n,i)=1$ for $i=2^{k-1}$ and $i=2^k$, and

$$N_{r-1,1}(2^{k-1})=N_{r-1,1}(2^k)=r-1.$$

Theorem 6.2 gives us

$$N_{r,1}(n) = N_{r-1,1}(n) + 1 + 2(r-1) - 2(r-1) = N_{r-1,1}(n) + 1.$$

Thus, if n is given by (6.3), we have

$$(6.4) \qquad N_{r,1}(n) = r, \quad N_{r,3}(n) = r^2 - r.$$

Now consider

$$(6.5) \qquad n = 2^s + 2^{k-1} + 2^k \quad (0 \le s < k-2).$$

Then, using [1] and (6.4), we have

$$\begin{aligned} f_1(n,i) &= 1 && \text{for } i = 2^s \text{ and } i = 2^{k-1} + 2^k, \\ f_3(n,i) &= 1 && \text{for } i = 2^{k-1}, 2^k, 2^s + 2^{k-1}, \text{ and } 2^s + 2^k, \\ N_{r-1,1}(x) &= \begin{cases} (r-1)^2 & \text{for } x = 2^s + 2^{k-1} \text{ and } x = 2^s + 2^k, \\ r-1 & \text{for } x = 2^{k-1} + 2^k, 2^s, 2^{k-1}, \text{ and } 2^k. \end{cases} \end{aligned}$$

Theorem 6.2 gives us $N_{r,1}(n) = N_{r-1,1}(n) + 2r - 1$. Thus, if n is given by (6.5), we have

$$N_{r,1}(n) = r^2, \quad N_{r,3}(n) = r^3 - r^2.$$

Using this method on the other cases of $B(n) = 3$, we can prove the following.

Theorem 6.3: Suppose $B(n) = 3$ and $D(n) > 0$. Then

$$N_r(n) = \begin{cases} \left(r^2, C(n)\binom{r}{2}r^2 + \binom{r}{3}r, \; r^3 - r^2\right) & \text{if } D(n) = 1, \\ \left(r^3 - 2r^2 + 2r, C(n)\binom{r}{2}r^2 + 2\binom{r}{3}r, \; 2r^2 - 2r\right) & \text{if } D(n) = 2. \end{cases}$$

We could next look at the case $B(n) = 4$ and get similar results. In general, after examining the case $B(n) = j$, we can move to the case $B(n) = j + 1$. As j increases, the formulas become much more complicated.

7. MULTINOMIAL COEFFICIENTS MODULO P

Let p be an odd prime and recall that $N_{r,m}^{(p)}(n)$ is the number of multinomial coefficients $(n_1, n_2, \ldots, n_r)$ such that $(n_1, n_2, \ldots, n_r) \equiv m \pmod{p}$.

Let n have base p expansion (5.2) and let A_j be the number of coefficients a_i $(0 \le a_i \le k)$ that are equal to j.

We shall use the definitions of t and $t(m)$ given at the beginning of section 4.

Theorem 7.1: Let n have expansion (5.2) and suppose that $0 \le a_i \le 2$ for each a_i. Let m be a positive integer.

(a) If there are no solutions to $2^x \equiv m \pmod{p}$, then $N_{r,m}^{(p)}(n) = 0$.

(b) If there are solutions to $2^x \equiv m \pmod{p}$, then

$$(7.1)\qquad N_{r,m}^{(p)}(n)=r^{A_1}\sum_{j=0}^{s}\binom{A_2}{t(m)+jt}\binom{r}{2}^{t(m)+jt}r^{A_2-t(m)-jt},$$

where $t(m)+st\le A_2<t(m)+(s+1)t$.

Proof: Suppose $(n_1,n_2,\ldots,n_r)\equiv m\ (\mathrm{mod}\ p)$. Since $m>0$, in Lemma 5.1 we must have $\varepsilon_i=0$ for $i=0,1,\ldots,k-1$. In (6.2) we see that each binomial coefficient on the right side will be congruent to 2^w modulo p for some $w\ge 0$, and we must have

$$(7.2)\qquad h=\Sigma(w)\ \text{ and }\ 2^h\equiv m\ \ (\mathrm{mod}\ p).$$

Thus, part (a) is clear. We now count the number of ways (7.2) can happen. Pick h of the A_2 rows adding up to 2, and pick two positions in each of these rows for 1's. There are $\binom{A_2}{h}\binom{r}{2}^h$ ways of doing this. In the remaining A_2-h rows, pick one position in each row for a 2. There are r^{A_2-h} ways of doing this. We see from the last part of Lemma 2.2 that when the binomial coefficients on the right side of (6.2) are broken down in terms of their coefficients modulo p, then we have

$$(n_1,n_2,\ldots,n_r)\equiv 2^h\equiv m\ \ (\mathrm{mod}\ p).$$

As we saw in the proof of Theorem 4.1, $h=t(m)+jt$ for some j, and (7.1) follows. This completes the proof.

Corollary: Let $p=3$. Then

$$N_{r,1}^{(3)}(n)=\frac{1}{2}\cdot r^{A_1}\left[\left(r+\binom{r}{2}\right)^{A_2}+\left(r-\binom{r}{2}\right)^{A_2}\right],$$

$$N_{r,2}^{(3)}(n)=\frac{1}{2}\cdot r^{A_1}\left[\left(r+\binom{r}{2}\right)^{A_2}-\left(r-\binom{r}{2}\right)^{A_2}\right].$$

We now prove a theorem analogous to Theorem 6.2. It follows immediately from

$$(n_1,n_2,\ldots,n_r)=\binom{n}{n_1}(n_2,\ldots,n_r).$$

Theorem 7.2 let m be a positive integer and suppose $\binom{n}{j}\not\equiv 0\ \ (\mathrm{mod}\ p)$. Let $g(j)$ be the smallest positive integer such that $\binom{n}{j}\cdot g(j)\equiv m\ \ (\mathrm{mod}\ p)$. Then

$$N_{r,m}^{(p)}(n)=\sum_j N_{r-1,g(j)}^{(p)}(n-j),$$

where the sum is over all j such that $0\le j\le n$ and $\binom{n}{j}\not\equiv 0\ \ (\mathrm{mod}\ p)$.

If n has the base p expansion (5.2), then in Theorem 7.2 the sum is over all j such that

$$j=\sum_{i=0}^{k}e_ip^i\quad (0\le e_i\le a_i).$$

For example, let $p=5$, $r=3$, $n=11=1+2\cdot 5$. Then

$$\begin{aligned}N_{3,1}^{(5)}(11) &= N_{2,1}^{(5)}(11)+N_{2,1}^{(5)}(10)+N_{2,3}^{(5)}(6)+N_{2,3}^{(5)}(5)+N_{2,1}^{(5)}(1)+N_{2,1}^{(5)}(0)\\ &= 4+2+0+0+2+1=9.\end{aligned}$$

Similarly, we can show that $N_{3,2}^{(5)}(11)=9$.

Theoretically, then, if we know the values of $N_{2,m}^{(p)}(n)$, we can use Theorem 7.2 to find $N_{r,m}^{(p)}(n)$ for any r.

For completeness, we can use (5.3) to obtain $N_{r,m}^{(2)}(1)=r^{B(n)}$.

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