ON EXTENDED GENERALIZED STIRLING PAIRS

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1. INTRODUCTION

Following Carlitz's terminology (see [2] and [3]), we define a generalized Stirling pair (GSP) as follows:

Definition 1: Let f and g belong to the commutative ring of formal power series with real or complex coefficients and let

(1) $$\sum_{n\geq 0} A_1(n,k)t^n / n! = (f(t))^k / k!$$

(2) $$\sum_{n\geq 0} A_2(n,k)t^n / n! = (g(t))^k / k!$$

Then $\{A_1(n,k),\ A_2(n,k)\}$ is called a GSP if and only if f and g are reciprocal (inverse) of each other, in the sense that

$$f(g(t)) = g(f(t)) = t \text{ with } f(0) = g(0) = 0.$$

From Carlitz [1], we have that $\{A_1(n,k),\ A_2(n,k)\}$ is a GSP if and only if the double sequences of numbers $A_1(n,k)$ and $A_2(n,k)$ satisfy the orthogonality relations

(3) $$\sum_{n=k}^{m} A_1(m,n)A_2(n,k) = \sum_{n=k}^{m} A_2(m,n)A_1(n,k) = \delta_{mk},$$

where δ_{mk} is the Kronecker symbol, or, equivalently, they satisfy the inverse relations

(4) $$a_n = \sum_{k=0}^{n} A_1(n,k)b_k,\quad b_k = \sum_{k=0}^{n} A_2(n,k)a_k,$$

where $n = 0, 1, 2, \ldots,$ and either $\{a_k\}$ or $\{b_k\}$ is given arbitrarily. That is, a GSP is characterized by a pair of orthogonality or inverse relations.

Note that the pair of Stirling numbers of the first and second kind is a special case of a GSP $[g(t) = e^t - 1,\ f(t) = \ln(1+t)]$.

In the present paper, we define an extended generalized Stirling pair, say $\{B_1(n,k),\ B_2(n,k)\}$, which covers in particular the above known results and other interesting pairs of numbers with combinatorial interpretations. Moreover, similar relations to the orthogonality ones are established which characterize this extended generalized Stirling pair.

Finally, recurrence and congruence relations concerning the numbers $B_i(n,k),\ i = 1,2,$ are obtained.

It is worth mentioning the following result, due to Carlitz (see [2]), which also leads to orthogonality relations. Let $\{f_k(z)\}$ denote a sequence of polynomials such that

$$\deg f_k(z) = k;\ f_k(0) = 0 \text{ for } k > 0;\ \left(1 + \sum_{n=1}^{\infty} c_n x^n / n!\right)^z = \sum_{k=0}^{\infty} f_k(z)x^k / k!$$

Put

$$F_1(n,n-k)=\binom{k-n}{k}f_k(n);\quad F(n,n-k)=\binom{n}{k}f_k(-n+k).$$

Then

$$\sum_{k=j}^{n}(-1)^{n-k}F_1(n,k)F(k,j)=\sum_{k=j}^{n}(-1)^{k-j}F(n,k)F_1(k,j)=\delta_{nj}.$$

The functions $F_1(z,z-k)$, $F(z,z-k)$ are called a Stirling pair, and a generalization is given in [3]. Note that for $f_k(z)$ the Norlund polynomial $B_k(z)$ defined by

$$\left(\frac{x}{e^x-1}\right)^z=\sum_{k=0}^{\infty}B_k(z)x^k/k!$$

where z is an arbitrary complex number (see [9, ch. 6]), the numbers $F_1(n,n-k)$ and $F(n,n-k)$ reduce to the ordinary Stirling numbers of the first kind (signless) and the ones of the second kind, respectively.

2. THE DEFINITION OF THE $\{B_1(n,k),\ B_2(n,k)\}$-SPECIAL CASES

We define an extended generalized Stirling pair (EGSP) as follows:

Definition 2: Let h, f, and g belong to Γ and let

(5) $$\sum_{n\geq 0}B_1(n,k)t^n/n!=h(t)(f(t))^k/k!,\quad h(t)\neq 0,$$

(6) $$\sum_{n\geq 0}B_2(n,k)t^n/n!=\frac{1}{h(g(t))}(g(t))^k/k!$$

Then $\{B_1(n,k),\ B_2(n,k)\}$ is called an extended, generalized Stirling pair (EGSP) if and only if f and g are reciprocal of each other.

Note that if $\{B_1(n,k),\ B_2(n,k)\}$ is an EGSP, then $B_i(n,k)=0,\ i=1,2$ if $k>n$. For $h(t)=1$, the pair $\{B_1(n,k),\ B_2(n,k)\}$ reduces to the one $\{A_1(n,k),\ A_2(n,k)\}$. Some interesting special cases of extended generalized Stirling pairs are given below.

1. For $h(t)=e^t$, $f(t)=t$, $g(t)=t$, we have the pair

$$\left\{\binom{n}{k},\ (-1)^{n-k}\binom{n}{k}\right\}.$$

2. For $h(t)=e^{\lambda t}$, λ a real number, $f(t)=e^t-1$, $g(t)=\ln(1+t)$, we have the pair

$$\{(-1)^{n-k}R_1(n,k,\lambda),\quad R(n,k,\lambda)\}$$

where $R_1(n,k,\lambda)$ and $R(n,k,\lambda)$ are the weighted Stirling numbers of the first and second kind, respectively (see [5]). Since $R_1(n,k,-\alpha)=(-1)^{n-k}s_\alpha(n,k)$ and $R(n,k,-\alpha)=S_\alpha(n,k)$ where $s_\alpha(n,k)$ and $S_\alpha(n,k)$ are the noncentral Stirling numbers of the first and second kind, respectively (see [8]), we have that $\{s_\alpha(n,k),\quad S_\alpha(n,k)\}$ is also an EGSP.

3. For $h(t)=(1+t)^s$, s a real number, $f(t)=(1+t)^r-1$, r a real number, $r\neq 0$, $g(t)=(1+t)^{\frac{1}{r}}-1$, we have the pair $\{G(n,k,r,s),\ G(n,k,1/r,-s/r)\}$. When both r and s are positive or negative integers, we have that $\frac{k!}{n!}G(n,k,r,s)$ is the number of ways of putting n like balls into k different cells of r different compartments each and a (control) cell of s different compartments with limited or unlimited capacity (see [6]).
4. For $h(t)=(1-t)^{\theta-\lambda}$, θ,λ real numbers, $f(t)=[1-(1-t)^{\theta}]/\theta$ and $g(t)=(1+\theta t)^{\mu}-1$ where $\mu\theta=1$, we have the pair $\{(-1)^{n-k}S_1(n,k,\lambda+\theta|\theta),\ S(n,k,\lambda|\theta)\}$ where $S_1(n,k,\lambda|\theta)$ and $S(n,k,\lambda|\theta)$ are the degenerate weighted Stirling numbers of the first and second kind, respectively (see [7]).

   Letting $\lambda=0$, we see that the degenerate numbers of Carlitz [4]

$$\{(-1)^{n-k}S_1(n,k|\theta),\quad S(n,k|\theta)\}$$

form a GSP, since $S_1(n,k,\theta|\theta)=S_1(n,k|\theta)$.

Letting $\theta=0$, we see again that $\{(-1)^{n-k}R_1(n,k,\lambda),\quad R(n,k,\lambda)\}$ is an EGSP.

3. CHARACTERIZATIONS

An EGSP is characterized by a pair of orthogonality relations, as we show in what follows.

Theorem 1: The numbers $B_i(n,k)$, $i=1,2$, are given by the relations (5) and (6), respectively. Then, $\{B_1(n,k),\ B_2(n,k)\}$ is an EGSP if and only if the following orthogonality relations,

$$\sum_{n=k}^{m}B_2(m,n)B_1(n,k)=\sum_{n=k}^{m}B_1(m,n)B_2(n,k)=\delta_{mk}, \tag{7}$$

hold, where δ_{mk} is the Kronecker symbol.

Proof: Setting $t\to g(t)$ in (5) and using (6), we get

$$\begin{aligned}(f(g(t)))^k/k! &= \sum_{n=k}^{\infty}B_1(n,k)\sum_{m=n}^{\infty}B_2(m,n)t^m/m!\\ &= \sum_{m=k}^{\infty}\left\{\sum_{n=k}^{m}B_2(m,n)B_1(n,k)\right\}t^m/m!\end{aligned} \tag{8}$$

Similarly, setting $t\to f(t)$ in (6) and using (5), we get

$$\frac{h(t)}{h(g(f(t)))}(g(f(t)))^k/k!=\sum_{m=k}^{\infty}\left\{\sum_{n=k}^{m}B_1(m,n)B_2(n,k)\right\}t^m/m! \tag{9}$$

The "if" part: Substituting the relations (7) into relations (8) and (9), we have

$$(f(g(t)))^k=t^k,\quad k=0,1,2,\dots,$$

and

$$\frac{h(t)}{h(g(f(t)))}(g(f(t)))^k=t^k,\ k=0,1,2,\dots,$$

from which we deduce that

$$f(g(t)) = g(f(t)) = t.$$

The "only if" part: Suppose that $f(g(t)) = g(f(t)) = t$ from relations (8) and (9), then we have

$$t^k / k! = \sum_{m=k}^{\infty} \left\{ \sum_{n=k}^{m} B_2(m,n) B_1(n,k) \right\} t^m / m!$$

$$= \sum_{m=k}^{\infty} \left\{ \sum_{n=k}^{m} B_1(m,n) B_2(n,k) \right\} t^m / m!$$

Equating coefficients of $t^m / m!$, we obtain the relations (7).

By the following theorem, we show that an EGSP is characterized by a pair of relations similar to the orthogonality relations.

Theorem 2: The numbers $B_i(n,k)$, $i = 1, 2$, are given by (5) and (6), respectively, and the numbers $A_i(n,k)$ $i = 1, 2$, are given by (1) and (2), respectively. Thus, $\{B_1(n,k),\ B_2(n,k)\}$ is an EGSP if and only if

$$(10) \qquad \sum_{n=k}^{m} B_1(m,n) A_2(n,k) = \binom{m}{k} h_{m-k}$$

where h_j, $j = 0, 1, 2, \ldots$, are the coefficients in the expansion

$$h(t) = \sum_{j=0}^{\infty} h_j t^j / j!$$

or

$$(11) \qquad \sum_{n=k}^{m} B_2(m,n) A_1(n,k) = \binom{m}{k} h^*_{m-k}$$

where h^*_j, $j = 0, 1, 2, \ldots$, are the coefficients in the expansion

$$\frac{1}{h(g(t))} = \sum_{j=0}^{\infty} h^*_j t^j / j!.$$

Proof: From relation (5), we get

$$\sum_{m=k}^{\infty} B_1(m,k) t^m / m! = \sum_{j=0}^{\infty} h_j t^j / j! \sum_{r=k}^{\infty} A_1(r,k) t^r / r!$$

$$= \sum_{m=k}^{\infty} \left\{ \sum_{j=0}^{m-k} \binom{m}{j} h_j A_1(m-j, k) \right\} t^m / m!$$

Equating coefficients of $t^m / m!$, we obtain

$$B_1(m,k) = \sum_{j=k}^{m} \binom{m}{j} h_{m-j} A_1(j, k).$$

Multiplying both sides of the above relation by $A_2(n,k)$ and summing for all $n=k,\ k+1,\dots,m$, we obtain

$$\sum_{n=k}^{m} B_1(m,n)A_2(n,k) = \sum_{n=k}^{m}\sum_{j=n}^{m}\binom{m}{j}h_{m-j}A_1(j,n)A_2(n,k). \tag{12}$$

The "if" part: Comparing the relations (12) and (10), we get

$$\binom{m}{k}h_{m-k} = \sum_{j=k}^{m}\binom{m}{j}h_{m-j}\sum_{n=k}^{j}A_1(j,n)A_2(n,k).$$

Multiplying both sides by $t^m/m!$ and summing for all $m=k,\ \ k+1,\dots$, we have

$$\frac{t^k}{k!}h(t) = \sum_{m=k}^{\infty}\sum_{j=k}^{m}\sum_{n=k}^{j}\binom{m}{j}h_{m-j}A_1(j,n)A_2(n,k)t^m/m!$$

$$= \sum_{j=k}^{\infty}\sum_{m=j}^{\infty}\sum_{n=k}^{j}A_1(j,n)A_2(n,k)\binom{m}{j}h_{m-j}t^m/m!$$

or

$$\frac{t^k}{k!} = \sum_{j=k}^{\infty}\sum_{n=k}^{j}A_1(j,n)A_2(n,k)\frac{t^j}{j!}$$

from which we obtain

$$\sum_{n=k}^{j}A_2(j,n)A_1(n,k) = \delta_{jk}.$$

Consequently, $\{A_1(n,k),\ \ A_2(n,k)\}$ is a GSP or, equivalently, $f(g(t)) = g(f(t)) = t$.

The "only if" part We have that $f(g(t)) = g(f(t)) = t$ or, equivalently, the relations (3) hold. Consequently, relation (12) becomes

$$\sum_{n=k}^{m}B_1(m,n)A_2(n,k) = \sum_{j=k}^{m}\binom{m}{j}h_{m-j}\delta_{jk} = \binom{m}{k}h_{m-k}.$$

Remark 1: The relations (10) and (11) lead to the inverse relations

$$A_2(m,k) = \sum_{n=k}^{m}\binom{n}{k}h_{n-k}B_1(m,n),$$

$$A_1(m,k) = \sum_{n=k}^{m}\binom{n}{k}h^*_{n-k}B_2(m,n).$$

4. RECURRENCES

Let $\{B_1(n,k),\ \ B_2(n,k)\}$ be an EGSP.

Differentiating both sides of relation (5) and of relation (6) with respect to t, we can easily obtain the following recurrences:

(13) $$B_1(n+1,k)=\sum_{j=0}^{n}\binom{n}{j}\alpha_{j+1}B_1(n-j,k)+\sum_{j=0}^{n}\binom{n}{j}f_{j+1}B_1(n-j,k-1)$$

where $f_j,\ j=0,1,\ldots,$ and $\alpha_j,\ j=0,1,\ldots,$ are, respectively, the coefficients in the expansions

$$f(t)=\sum_{j=1}^{\infty}f_jt^j/j! \text{ and } h'(t)/h(t)=\sum_{j=0}^{\infty}\alpha_jt^j/j!,\ \ h'(t)=\frac{d}{dt}h(t)$$

and

(14) $$B_2(n+1,k)=\sum_{j=0}^{n}\binom{n}{j}\beta_{j+1}B_2(n-j,k)+\sum_{j=0}^{n}\binom{n}{j}g_{j+1}B_2(n-j,k-1)$$

where $g_j,\ j=0,1,\ldots,$ and $\beta_j,\ j=0,1,\ldots,$ are, respectively, the coefficients in the expansions

$$g(t)=\sum_{j=1}^{\infty}g_jt^j/j! \text{ and } \left(\frac{1}{\mathrm{h(g)(t))}}\right)'\Bigg/\left(\frac{1}{h(g(t))}\right)=\sum_{j=0}^{\infty}\beta_jt^j/j!.$$

Remark 2: Let $\{B_1(n,k),\ B_2(n,k)\}$ be an EGSP. From the definition of Bell polylnomials (cf. Riordan [10]),

$$Y_n(gf_1,gf_2,\ldots,gf_n)=\sum\left(\frac{n!g_k}{j_1!j_2!\ldots j_n!}\left(\frac{f_1}{1!}\right)^{j_1}\left(\frac{f_2}{2!}\right)^{j_2}\cdots\left(\frac{f_n}{n!}\right)^{j_n}\right)$$

where the sum extends over all n-tuples $(j_1,j_2,\ldots j_n)$ of nonnegative integers such that $j_1+j_2+\cdots+j_n=k$ and $j_1+2j_2+\cdots+nj_n=n$, one may obtain the following system of linear equations,

(15) $$y_n(gf_1,\ldots,gf_n)=\delta_{n1}\quad(n=1,2,\ldots),$$

from which we conclude that, for any given sequence $\{f_j\}$, the sequence $\{g_j\}$ can be determined.

We also have

$$\begin{aligned}\left(\frac{1}{h(g(t))}\right)'\Bigg/\left(\frac{1}{h(g(t))}\right)&=-(h(g(t))'/h(g(t))=-g(t)\sum_{i\ge0}h_{i+1}\frac{(g(t))^i}{i!}\ \frac{1}{h(g(t))}\\&=-\sum_{s\ge0}g_{s+1}\frac{t^s}{s!}\sum_{i\ge0}h_{i+1}\sum_{r=i}^{\infty}B_2(r,i)\frac{t^r}{r!}\\&=-\sum_{j\ge0}\left\{\sum_{r=0}^{j}\sum_{i=0}^{r}h_{i+1}B_2(r,i)g_{j-r+1}\binom{j}{r}\right\}\frac{t^j}{j!}\end{aligned}$$

from which we get

(16) $$\beta_j=\sum_{i=0}^{j}h_{i+1}\sum_{r=i}^{j}\binom{j}{r}g_{j-r+1}B_2(r,i),\quad j=0,1,\ldots\ .$$

From relations (15) and (16) we have that, for any given sequences $\{f_i\}$ and $\{h_j\}$, the sequences $\{g_j\}$ and $\{\beta_j\}$ can be determined.

Consequently, having the recurrence (13), we may conclude the one (14).

An interesting special case of the above situation is given by the following Proposition.

Proposition 1: Let $\{B_1(n,k),\ B_2(n,k)\}$ be an EGSP and the numbers $B_1(n,k)$ satisfy the triangular array recurrence relation

$$B_1(n+1,k)=(c_1n+c_2k+c_3)B_1(n,k)+c_4B_1(n,k-1) \tag{17}$$

where c_i, $i=1,2,3,4$ constants, $k=0,1,\dots,n+1$, $n=0,1,2,\dots$, with initial conditions

$$B_1(0,0)=1,\quad B_1(n,k)=0 \text{ if } n<k,$$

then the numbers $B_2(n,k)$ satisfy the triangular array recurrence relation

$$B_2(n+1,k)=\left(-\frac{c_2}{c_4}n-\frac{c_1}{c_4}k-\frac{c_3}{c_4}\right)B_2(n,k)+\frac{1}{c_4}B_2(n,k-1) \tag{18}$$

where $k=0,1,\dots,n+1$, $n=0,1,\dots$, with initial conditions

$$B_2(0,0)=1,\quad B_2(n,k)=0 \text{ if } n<k.$$

Proof: Multiplying both sides of (17) by $t^n/n!$, summing for all $n=0,1,\dots$, and using relation (5), we have that (17) holds if and only if

$$h'(t)=\frac{c_3h(t)}{1-c_1t} \quad \text{and} \quad f'(t)=\frac{c_2f(t)+c_4}{1-c_1(t)}. \tag{19}$$

Moreover,

$$\left(\frac{1}{h(g(t))}\right)'=\frac{-c_3(1/h(g(t)))}{c_2t+c_4}$$

and

$$g'(t)=1/f'(g(t))=\frac{(-c_1/c_4)g(t)+1/c_4}{(c_2/c_4)t+1},$$

which, on using (6) and (19), gives (18).

5. CONGRUENCES

Let $\{B_1(n,k),\ B_2(n,k)\}$ be an EGSP and $\{A_1(n,k),\ A_2(n,k)\}$ be the corresponding GSP given by Definitions 2 and 1, respectively. In this section we are interested in integer pairs. A question now arises: Under what conditions on $h(t)$ and $f(t)$ are the above numbers integers?

Supposing that $h(t)$ and $f(t)$ are Hurwitz series, in the sense that the coefficients h_j, $j=0$, 1, ..., and f_j, $j=0$, 1, ..., in their expansions are integers, and that $h(0)=f(0)=1$; it can easily be proved, on using Taylor's expansions and the relation $f(g(t))=t$, that $g(t)$ and $1/h(g(t))$ are also Hurwitz series. In this case, the numbers $A_i(n,k)$, $i=1,2$, and $B_i(n,k)$, $i=1,2$, are integers, as we can easily see from their definitions and the fact that if $f(0)=0$ and $f(t)$ is a Hurwitz series, then $(f(t))^k/k!$, $k=0,1,\dots$, is also a Hurwitz series, and that Hurwitz series are closed under multiplication.

We have already proved the following Proposition.

Proposition 2: Let $\{B_1(n,k),\ B_2(n,k)\}$ be an EGSP and $\{A_1(n,k),\ A_2(n,k)\}$ be the corresponding GSP. If $h(t)$ and $f(t)$ are Hurwitz series with $h(0)=f'(0)=1$, then $B_i(n,k),\ i=1,2$, and $A_i(n,k),\ i=1,2$, are integers.

Now, from the proof of Theorem 2, we have

$$B_1(m,k)=\sum_{j=k}^{m}\binom{m}{j}h_{m-j}A_1(j,k),\ \ k=1,2,\dots,m \tag{20}$$

and, similarly,

$$B_2(m,k)=\sum_{j=k}^{m}\binom{m}{j}h^*_{m-j}A_2(j,k),\ \ k=1,2,\dots,m. \tag{21}$$

Using (20) and (21) and the fact that $\binom{p}{j}\equiv 0 \pmod p$ for each prime p, except $\binom{p}{0}=\binom{p}{p}=1$, we obtain the following congruence:

$$B_i(p,k)\equiv A_i(p,k) \pmod p,\ \ i=1,2, \tag{22}$$

for each prime p, $k=1,2,\dots,p$, while $B_1(p,0)=h_p$, $B_2(p,0)=h^*_p$ and $A_i(p,0)=0$, $i=1,2$.

Also, using relations (7) and

$$B_1(m,m)=B_2(m,m)=1,$$

we obtain

$$B_2(m,k)=-\sum_{j=k}^{m-1}B_1(m,j)B_2(j,k)$$

and

$$B_1(m,k)=-\sum_{j=k}^{m-1}B_2(m,j)B_1(j,k)$$

from which, on using (22) and (10), we get

$$B_i(p,k)\equiv A_i(p,k)\equiv 0 \pmod p,\ \ i=1,2,$$

for each prime p, $k=1,2,\dots,p-1$.

As examples of integer EGSP's we give, using Proposition 2, the special cases of EGSP's referred to in the present work (§2) for λ, s, θ integers and $r=\pm 1$.

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GENERALIZED PASCAL TRIANGLES AND PYRAMIDS THEIR FRACTALS, GRAPHS, AND APPLICATIONS

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As stated by the author in his preface, this monograph is devoted to the more profound questions connected with the study of the Pascal triangle, and its planar as well as spatial analogs. It also contains an extensive discussion of the divisibility of the binomial, trinomial, and multinomial coefficients by a prime p, as well as the distributions of these coefficients with respect to the modulus p, or p^s, in corresponding arithmetic triangles, pyramids and hyperpyramids. Particular attention is also given to the subject of fractals obtained from the Pascal triangle and other arithmetic triangles. The author also constructs and investigates matrices and determinants whose elements may be binomial, generalized binomial and trinomial coefficients, and other special values. Furthermore, the author pays particular attention to the development of effective combinatorial methods and algorithms for the construction of basis systems of polynomial solutions of partial differential equations, including equations of higher order and with mixed derivatives. The algorithms he proposes are invariant with respect to the order, and the iteration, of operators arising in connection with the differential equations. Finally, the author also discusses non-orthogonal polynomials of binomial type, and polynomials whose coefficients may be Fibonacci, Lucas, Catalan, and other special numbers.

The monograph first published in Russia in 1990 consists of seven chapters, a list of 406 references, an appendix with another 126 references, many illustrations and specific examples. Fundamental results in the book are formulated as theorems and algorithms or as equations and formulas.

The intention of the translator is to make the work of Dr. Bondarenko widely accessible because he feels that Dr. Bondarenko has done the mathematical community a valuable service by writing a useful and interesting compendium of results on Pascal's triangle as well as its ramifications.

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