CONNECTIONS IN MATHEMATICS: AN INTRODUCTION TO FIBONACCI VIA PYTHAGORAS

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1. INTRODUCTION

We are all familiar with the traditional presentation of the Fibonacci sequence in classes designed for liberal arts majors. We "wow" the students with pinecones and pineapples! We talk to them about the wondrous "appearance" of Fibonacci numbers in **their** world. But can we use Fibonacci to bring these students into **our** world?

Is it possible for liberal arts majors to appreciate mathematics—apart from applications of the subject in art, nature, and other areas? Can we develop courses that instill in students a sense of excitement about making connections in mathematics, given the attitudes toward our subject that many of them bring to class? As Lynn Arthur Steen says: "For students in the arts and humanities, mathematics is an invisible culture—feared, avoided, and consequently misunderstood" (3).

Designing requisite[1] mathematics courses for liberal arts students is difficult. In attempting to give them a sense of mathematics, we often resort to overstressing its utility—reaching for areas outside of mathematics to validate the study of the subject. Or we assemble what appears to the students a collection of disjoint topics—offering little motivation for them to search for connections.

The challenge is to draw the students **into** mathematics—generating in them an excitement about making mathematical connections and an appreciation of fundamental interrelationships between topics. I believe I have met this challenge by introducing Fibonacci via Pythagoras, and I want to share the experience![2]

2. RATIONALE

Mathematics builds upon itself in a way that other sciences do not. Even topics developed in antiquity continue to be relevant today to mathematical growth—knowledge gives rise to new knowledge; problems generate new problems. One objective in teaching mathematical concepts is to give students a sense of how these ideas fit into the edifice we call mathematics. Examination of connections between mathematical topics is one way to achieve this goal.

The connection between Fibonacci numbers and Pythagorean triples is well known (cf. [6], [16], [17]). But this connection is not frequently used to *introduce* Fibonacci numbers. I propose a classroom lesson that involves students working with a familiar topic (Pythagorean triples) prior to connecting it to an unfamiliar one (Fibonacci numbers). In my experience, the preliminary work with triples motivates a discussion of this connection, and stimulates students to want to

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[1]In 1983, the largest public university system in the country—the California State University System—established a mathematics course as a graduation requirement for all students at any of its nineteen campuses.

[2]The appendix includes an annotated list of books, journal articles, and videotapes that I used as resources for classroom discussion and projects. All numbers in brackets [ ] refer to the resources listed in this appendix.

learn more about the Fibonacci numbers. Perhaps more importantly, it creates in the students a genuine excitement about making **mathematical** connections.

As an alternative to what is perhaps a traditional introduction to the Fibonacci numbers based on their "surprise" applications in nature—breeding rabbits, patterns in pinecones and pineapples, etc. (cf. [9], [12], [19]), my presentation enables students to experience this kind of "surprise" in connecting **mathematical** areas. The classroom experience begins with discussion of problems inspired by Pythagorean triples incorporating assignments and activities generating Pythagorean triples; then follows with an examination of connections between the products of Fibonacci and Pythagoras and an investigation of the historical and present-day significance of the Fibonacci numbers. This approach conforms to the goals expressed by Alvin White (1985) in his article "Beyond Behavioral Objectives":

> . . . our guidelines and teaching objectives should not have as their major target or focus the mastery of facts and techniques. Rather the facts and techniques should be the skeletal framework which supports our objective of imbuing our students with the spirit of mathematics and a sense of excitement about the historical development and the creative process. (3, p. 850)

3. CLASSROOM LESSONS

A. Pythagorean Triples

I begin by proposing that Pythagorean triples give evidence of how problems generate new problems in mathematics. The students are amazed to discover that such integers were known in ancient civilizations 1200 years before Pythagoras. I try to give them a sense that what perhaps is more exciting is that these triples have inspired interesting problems in mathematics since the time of Pythagoras. My students learn, by perusing the list of resources, that the generation of Pythagorean triples is a topic that still fascinates some contemporary mathematicians (cf. [2], [5], [6], [10], [11], [14], [16], [17], [20], [22], [23]).

I challenge the students to work in groups to discover common characteristics of Pythagorean triples with the goal of finding a generating form for them. I begin by asking the students to create triples, after we list the ones known to them. This induces a discussion of multiples of triples and a conjecture that multiples of triples are triples, motivating the need for a proof that this is indeed so. I form the class into small groups (four to five students per group) and ask them to form conjectures about characteristics of primitive Pythagorean triples [we had read and discussed Polya's heuristics for problem solving (1)]. The students work together, observing patterns and making guesses. For example, by looking at (3, 4, 5), (5, 12, 13), and (8, 5, 17), (12, 35, 37), they make the following conjectures: only one number of the triple could be even; when the smallest number of the triple is even, the difference between the two larger numbers is two; when the smallest number of the triple is odd, the difference between the two larger numbers is one; that five is a factor of some number of the triple; etc. Then, as a class, we discuss each group's conjectures, attempting to prove or disprove *their* hypotheses. Their conjectures introduce many interesting class discussions about numbers and their relationships. We explored, for example, questions of divisibility, prime factorization, what it means for integers to be relatively prime, etc. After the students play with the Pythagorean triples and examine some characteristics of these numbers, they are eager to find a systematic way to generate them.

Methods for obtaining Pythagorean triples cited in the Annotated List of Resources range from simple to sophisticated. I refer to several so the students can get a sense of the range of options (cf. [10], [14], [20], [23]). One they particularly like is Kalman's method (cf. [20]) for generating Pythagorean triples from proper fractions. Kalman starts with a right triangle with angle A, such that $\tan A$ = a proper fraction, say p/q. He then constructs another right triangle using $2A$ as one angle. Since $\tan 2A = 2\tan A / (1-\tan^2 A) = 2pq/(q^2-p^2)$, the legs of the new triangle can be labeled $2pq$ and (q^2-p^2). Using the Pythagorean Theorem to determine the length of the hypotenuse will produce an integer. This proves that Kalman's procedure always produces a Pythagorean triple when $\tan A$ is rational. With the "hands-on" experience of generating triples and the knowledge gained from exploring other attempts at generating triples using familiar objects (like fractions), the students are ready to venture into unfamiliar territory.

B. Fibonacci Numbers

Since the students have an understanding of the difficulties involved in generating Pythagorean triples and a look at the diversity of methods for doing so, they are ready to learn about the connection between the mathematical products of Fibonacci and Pythagoras. We read and discuss several articles describing the use of Fibonacci numbers to produce Pythagorean triples (cf. [6], [16], [17]). The students are intrigued with the connection. The discussion of Fibonacci and Pythagoras provides a historical perspective and the use of Fibonacci's numbers to generate Pythagoras' numbers illustrates how mathematics builds upon itself using newer techniques to re-examine old problems.

The videotape *The Theorem of Pythagoras* (cf. [2]) shows dynamical versions of dissection proofs of the Pythagorean theorem. For a classroom activity, I organize students into small groups to "play with" a cardboard model of a dissection proof, asking them to assemble pre-cut pieces to illustrate the proof. This gives them a sense of what is involved in a dissection proof. I then follow with a classroom experiment based on the idea of dissection proof designed to show the students that evidence is different from proof and to prepare the way for a Fibonacci connection. I ask the students to construct an 8×8 square and calculate the area of 64. Then I direct them to dissect their model into a 5×13 figure as indicated:

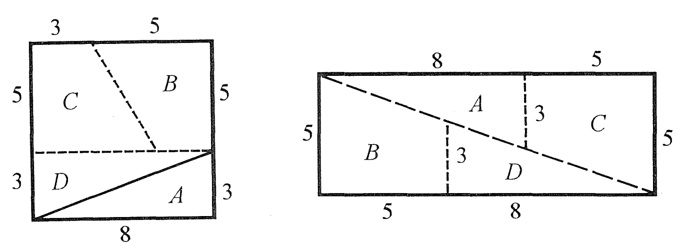

FIGURE 1

They quickly assume this figure is a rectangle, so when I ask them to calculate its area, they compute an area of 65. This seems to "prove" that 64 (the area of the original square) is equal to 65 (the area of the rectangle formed form the dissected pieces of the square). Our investigation of the new "rectangle" (via similar triangles) illustrates that the reconstructed figure is not truly a rectangle. This activity reinforces the necessity of rigorous proof in mathematics and alerts students to the dangers of accepting visual evidence as proof.

The culmination of this lesson is reading and discussing the article "Fibonacci Sequences and a Geometrical Paradox" (cf. [15]) in which Horadam shows how the Fibonacci numbers can be used to describe the area that appears to be gained in rearranging the parts of the square to form a rectangle. Using Horadam's article as a guide, we again analyze our 5×13 rectangle rearranged from the 8×8 square (Fig. 1). We observe that the one unit gain in area can be described by the relationship $5 \times 13 - 8^2 = 1$, a particular example of connecting three successive Fibonacci numbers (F_n, F_{n+1}, F_{n+2}) by the generalized formula $F_n F_{n+1} - F_{2n+1} = (-1)^{n+1}$. We then examine the relationship by considering the two cases, discovering: 1) when n is odd (as in our Fig. 1), the gain of one unit is the result of the appearance within the rectangle of a small parallelogram of unit area; 2) when n is even, the loss of one unit occurs because the unit parallelogram overlaps the dissected pieces. This gives the students visual evidence of how the Fibonacci numbers can be used to explain their dissection experiment, and how the results of their experiment can be expanded to include other cases. It is again, for them, yet another experience of utility **within** mathematics—the use of one mathematical topic to explain or clarify another.

Now the students, appreciating the use of Fibonacci numbers within mathematics, are ready to explore the many directions that Fibonacci numbers can inspire outside of mathematics —describing natural phenomena, determining outcomes of games, providing economic solutions for ecological problems, etc. (cf. List of Resources). Because the students were tuned in to the idea of connections, these discussions and activities were more meaningful than they had ever been in any previous classes I had taught.

4. CONCLUSION

My presentation of Fibonacci numbers via Pythagorean triples at the beginning of the course helps students to see that mathematical concepts often interrelate. The success of this classroom experiment lies in getting students to appreciate these interrelationships—enabling them to experience satisfaction in making mathematical connections. They learn to appreciate utility **within** mathematics as well as exterior to it. This experience set the tone for the entire course. One student wrote:

> I never learned interesting things like this in high school algebra. This topic contributed the most to my intellectual growth this semester, because it grabbed my attention, and allowed me to be open to other new concepts that we would study throughout the semester. The Fibonacci sequence opened the door to my mind, for it made me realize that math is going on all around me, and that it's important for me to understand why.

I encourage you to replicate this classroom experiment, and I welcome your reports about the results.

REFERENCES

1. George Polya. *How To Solve It: A New Aspect of Mathematical Method.* New Jersey: Princeton University Press, 1973.
2. Lynn Arthur Steen. "Restoring Scholarship to Collegiate Mathematics." *Focus* **6** (1986):1-7.
3. Alvin White. "Beyond Behavioral Objectives." *Amer. Math. Monthly* **82** (1985):849-51.

APPENDIX

ANNOTATED LIST OF RESOURCES

[1] Apostol, Tom. "The Pythagorean Theorem." In *MATHEMATICS!* California Institute of Technology, 1988.

This is an award-winning 20-minute computer-animated videotape, with accompanying workbook, on history, proofs, and applications of the theorem.

[2] Arpaia, P. J. "A Generating Property of Pythagorean Triples." *Mathematics Magazine* **44** (1971):26-27.

Based on the generating property of pairs of Pythagorean triples given by Courant and Robbins in *What Is Mathematics?*, this note establishes a generating property of any Pythagorean triple.

[3] Basin, S. L. "Generalized Fibonacci Sequences and Squared Rectangles." *American Mathematical Monthly* **70** (1963):372-79.

The author shows how generalized Fibonacci numbers can be used to generate squared rectangles (the problem of squaring a rectangle first appeared in the literature as a mathematical puzzle). The article concludes with an application (a ladder-network in communications systems) based on a model of the squaring of a rectangle of order n. This article is difficult, but not inaccessible to liberal arts majors.

[4] Beran, Ladislav. "Schemes Generating the Fibonacci Sequence." *Mathematical Gazette* **70** (1986):38-40.

The author shows that a resistance equation can be written in terms of the Fibonacci sequence and proves it by induction.

[5] Bergum, Gerald, & Yocom, Ken. "Tchebysheff Polynomials and Primitive Pythagorean Triples.," In *Two Year College Mathematics Readings*, Washington, D.C: The Mathematical Association of America, 1981.

This note illustrates and proves how to produce primitive integer-sides of a right triangle with hypotenuse c^n when the set of integers $\{a, b, c\}$ is primitive.

[6] Boulger, William. "Pythagoras Meets Fibonacci." *Mathematics Teacher* **82.4** (1989):277-81.

Boulger makes a nice connection between Fibonacci numbers and Pythagorean triples and proves it. He also illustrates a relationship between Fibonacci numbers and the golden ratio.

[7] DeTemple, Duane. "A New Angle on the Geometry of the Fibonacci Numbers." *Fibonacci Quarterly* **19.1** (1981):35-39.

The author gives a very nice geometric visualization of the Fibonacci sequence by representing the Fibonacci numbers by gnomons (Pythagorean carpenter squares).

[8] Federico, P. J. "A Fibonacci Perfect Squared Square." *American Mathematical Monthly* **71** (April 1964):404-06.

The author (following on Basin's paper [3]) describes how to construct "Fibonacci perfect squared squares."

[9] Freeman, W. H. (publisher). "Scale and Form." In *For All Practical Purposes*, 1988.

This is a 30-minute videotape that discusses Fibonacci numbers and gives applications in nature. This videotape supplements the mathematics textbook *For All Practical Purposes.*

[10] Goodrich, Merton. "A Systematic Method of Finding Pythagorean Numbers." *Mathematics Magazine* **19** (1945):395-97.

The author constructs a table to accomplish just what the title suggests.

[11] Hildebrand, W. J. "Generalized Pythagorean Triples." *The College Mathematics Journal* **16.1** (1985):48-52.

This article presents an algorithm to generate all possible integer triples (a,b,c) that are the sides of a triangle that contains angle C whose cosine is the rational number p/q. It also gives applications. The article assumes a knowledge of calculus and elementary number theory.

[12] Honsberger, Ross. "A Second Look at the Fibonacci and Lucas Numbers." In *Mathematical Gems III.* Washington, D.C.: The Mathematical Association of America, 1985.

This wonderfully written chapter demonstrates novel applications of Fibonacci numbers, proofs of many results, and sets of exercises for students. The author discusses the genealogy of the male honeybees and connections between Fibonacci and Lucas numbers.

[13] Honsberger, Ross. "The Fibonacci Sequence." In *Mathematical Morsels.* Washington, D.C.: The Mathematical Association of America, 1978.

The author discusses the problem of determining how many terms of the Fibonacci sequence are less than or equal to a given natural number N.

[14] Honsberger, Ross. "Pythagorean Arithmetic." In *Ingenuity in Mathematics.* Washington, D.C.: The Mathematical Association of America, 1970.

This short essay describes the arithmetic methods of the Pythagoreans and develops a procedure for generating Pythagorean triples. It is interesting reading for high school and college students and a good source of problems for instructors.

[15] Horadam, A. F. "Fibonacci Sequences and a Geometrical Paradox." *Mathematics Magazine* **35** (1962):1-11.

The author recalls the geometric paradox of subdividing an 8 unit square with area 64 square units and rearranging it to form a 5×13 unit rectangle with area 65 square units. He notes how the relationship $5 \times 13 - 8^2 = 1$ is a particular example of a result connecting 3 successive Fibonacci numbers. In the article, he extends the paradox to cover all sets of three consecutive Fibonacci numbers and generalizes the paradox by means of a generalized Fibonacci sequence.

[16] Horadam, A. F. "On Khazanov's Formulae." *Mathematics Magazine* **36** (1963):219-20.

The author outlines Khazanov's method for finding Pythagorean triples, and his own method for finding triples using a generalized Fibonacci sequence, then illustrates the connection between the two.

[17] Horadam, A. F. "Fibonacci Number Triples." *American Mathematical Monthly* **68** (1961): 751-53.

The author explores the connection between generalized Fibonacci numbers and Pythagorean triples.

[18] Howlett, G. et al. "Consecutive Heads and Fibonacci." *Mathematical Gazette* **69** (1985): 208-11.

These authors discuss the use of Fibonacci to determine the expected number of tosses of a coin before two consecutive heads are obtained. They comment on the Roland article "Toss Fibonacci" (see below).

[19] Jean, Roger, & Johnson, Marjorie. "An Adventure into Applied Mathematics with Fibonacci Numbers." *School Science and Mathematics* **89.6** (1989):487-98.

In this article, the authors illustrate how Fibonacci numbers arise in concrete situations: the genealogy of drones (bees), growth of sunflowers and pinecones, reflections on glass plates, economic solutions for the treatment of sewage in towns along a river bank, etc. They include a short bibliography with other resources of this type.

[20] Kalman, Dan. "Angling for Pythagorean Triples." *The College Mathematics Journal* **17.2** (1986):167-68.

The author generates Pythagorean triples from common fractions. A knowledge of trigonometry is assumed.

[21] Kuipers, L. posed the problem "No Fibonacci Pythagorean Triples." Solution given by M. Bicknell-Johnson in *The Fibonacci Quarterly* **27.1** (1989):88.

The solver proves that there are no positive integers r, s, and t, such that (F_r, F_s, F_t) is a Pythagorean triple.

[22] Nishi, Akihiro. "A Method of Obtaining Pythagorean Triples." *American Mathematical Monthly* **94.4** (1987):869-72.

This article, which develops what the title indicates, assumes a .knowledge of elementary number theory.

[23] Ore, Oystein. "The Pythagorean Problem." In *Invitation to Number Theory.* Washington, D.C.: the Mathematical Association of America, 1967.

This chapter introduces Pythagorean triples and develops the formula for generating them. It concludes with problems related to Pythagorean triangles. Mathematical skill at the level of intermediate algebra is sufficient to comprehend this presentation.

[24] Roland, Tim. "Toss Fibonacci." *Mathematical Gazette* **68** (1984):183-86.

The author gives some probability problems and makes what he calls "natural connections" with Fibonacci.

[25] Tomkins, A., & Pitt, D. "Runs and the Generalized Fibonacci Sequence." *Mathematical Gazette* **69** (1985):109-13.

This article provides a nice entry into statistics and recursive relationships. The authors tackle the question of winning in a gambling system by increasing the amount staked each time one loses. They ask: "In a given number of races, on how many occasions are we to expect a run of losers of a certain length?" Generalized Fibonacci sequences provide an answer.

[26] Vajda, S. *Fibonacci & Lucas Numbers, and the Golden Section: Theory and Applications* New York: Halsted Press, a division of John Wiley & Sons, 1989.

This book includes discussions in algebra, analysis, geometry, probability theory, and number theory. The author begins with a brief survey of problems that are solved by Fibonacci numbers, divisibility properties, generation of random numbers, game theory; the Golden Section and properties of platonic solids.

[27] Vorob'ev, N. N. *Fibonacci Numbers* (translation of *Chisla fibonachchi* [Moscow-Leningrad: Gostekhteoretizdat, 1951]). London: Pergamon Press, Ltd., 1961.

This book contains a set of problems that were the themes of several meetings of a mathematics club of Leningrad State University in 1949-1950. It is a wonderful resource for liberal arts classes as it rarely requires any knowledge beyond high school mathematics.

Subject Index

Mathematical Subject Classification

859-98 Mathematical education, collegiate

859-51 Geometry

859-01 History and Biography

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