ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems

PROBLEM PROPOSED IN THIS ISSUE

H-475 *Proposed by Larry Taylor, Rego Park, NY*

Professional chess players today use the algebraic chess notation. This is based upon the algebraic numbering of the chessboard. The eight letters *a* through *h* and the eight digits 1 through 8 are used to form sixty-four combinations of a letter and a digit which are called "symbol pairs." Those sixty-four symbol pairs are used to represent the sixty-four squares of the chessboard.

Develop a viable arithmetic numbering of the chessboard, as follows:

(a) Use twenty-five letters of the alphabet (all except *U*) and nine decimal digits (all except zero) to form 225 symbol pairs; choose sixty-four of those symbol pairs to represent the sixty-four squares of the chessboard.

(b) There are thirty-six squares from which a King can move to eight other squares. Let the nine symbol pairs representing the location of the King and the squares to which it can move contain all nine decimal digits.

(c) There are sixteen squares from which a Knight can move to eight other squares. A Queen located on one of those sixteen squares, moving one or two squares, can go to sixteen other squares. Let the twenty-five symbol pairs representing the location of the Knight or the Queen and the squares to which the Knight or the Queen can move contain all twenty-five letters of the alphabet.

(d) Let the algebraic Square *a*8 (the original location of Black's Queen Rook) correspond to the arithmetic Square *A*1; let the algebraic Square *h*1 (the original location of White's King Rook) correspond to the arithmetic Square *Z*9.

H-476 *Proposed by H.-J. Seiffert, Berlin, Germany*

Define the Pell numbers by $P_0 = 0$, $P_1 = 1$, $P_n = 2P_{n-1} + P_{n-2}$, for $n \geq 2$. Show that, for all positive integers n,

$$P_n = \sum_{\substack{k=0 \\ 4 \nmid 2n+k}}^{n-1} (-1)^{[(3k+3-2n)/4]} 2^{[3k/2]} \binom{n+k}{2k+1},$$

where [ ] denotes the greatest integer function.

<u>H-477</u> ***Proposed by Paul S. Bruckman, Edmonds, WA***

Let

$$F_r(z) = z^r - \sum_{k=0}^{r-1} a_k z^{r-1-k}, \tag{1}$$

where $r \geq 1$, and the a_k's are integers. Suppose F_r has distinct zeros θ_k, $k = 1, 2, \ldots, r$, and let

$$V_n = \sum_{k=1}^{r} \theta_k^n, \ n = 0, 1, 2, \ldots. \tag{2}$$

Prove that, for all primes p,

$$V_p \equiv a_0 \pmod{p}. \tag{3}$$

SOLUTIONS

Editorial Notes: *Paul S. Bruckman's name was omitted as a solver of H-435.*

A number of readers pointed out that exponent "u" was missing in two places in H-472.

Larry Taylor feels that the solution of H-454 as published was not complete, or at least was not what was intended. We therefore offer Mr. Taylor's solution here.

<u>Mix and Match</u>

<u>H-454</u> ***Proposed by Larry Taylor, Rego Park, NY***
(Vol. 29, no. 2, May 1991)

Construct six distinct Fibonacci-Lucas identities such that

(a) Each identity consists of three terms;
(b) Each term is the product of two Fibonacci numbers;
(c) Each subscript is either a Fibonacci or a Lucas number.

Solution by the proposer

Let j, k, n, and t be integers. It is known that $F_j F_{n+k} = F_k F_{n+j} - F_{k-j} F_n (-1)^j$.

(1) Let $j = F_t, k = F_{t+1}, n = F_{t-1}$;
(2) Let $j = F_t, k = L_{t+1}, n = F_{t-1}$;
(3) Let $j = F_{t-1}, k = F_t, n = F_{t+1}$;
(4) Let $j = F_{t-1}, k = L_t, n = L_{t+1}$;
(5) Let $j = F_{t-1}, k = F_{t+1}, n = F_t$;
(6) Let $j = F_{t-2}, k = F_{t+2}, n = F_t$.

In each of the six identities, each of $n+k, n+j, k-j$ is either a Fibonacci or a Lucas number.

<u>Simply Wonderful</u>

<u>H-458</u> ***Proposed by Paul S. Bruckman, Edmonds, WA***
(Vol. 29, no. 3, August 1991)

Given an integer $m \geq 0$ and a sequence of natural numbers $a_0, a_1, \ldots, a_m$, form the periodic simple continued fraction (s.c.f.) given by

$$\theta = [a_0; \overline{a_1, a_2, \ldots, a_2, a_1, 2a_0}]. \tag{1}$$

The period is symmetric, except for the final term $2a_0$, and may or may not contain a central term [that is, a_m occurs either once or twice in (1)]. Evaluate θ in terms of nonperiodic s.c.f.'s.

Solution by Russell Jay Hendel, Dowling College, Oakdale, NY

Let n denote the length of the period of θ. We claim

Theorem: $\theta^2 = a_0^2 + 2Ma_0 + N$, where, for $n \geq 5$,

$$M = [0; a_1, a_2, \ldots, a_2, a_1]$$
$$N / M = [0; a_1, a_2, \ldots, a_2].$$

Remark 1: For $n \leq 4$, we have:

$n = 4$:	$M = [0; a_1, a_2, a_1]$	$N / M = [0; a_1, a_2]$
$n = 3$:	$M = [0; a_1, a_1]$	$N / M = [0; a_1]$
$n = 2$:	$M = [0; a_1]$	$N / M = [0]$
$n = 1$:	$M = [0]$	$N = 1.$

These initial cases were verified using DERIVE. The study of the initial cases also aided discovery of the general pattern.

Remark 2: Results on the continued fractions of quadratic irrationals are well known. Some standard references are [2; pp. 310-88] or [3; pp. 197-204]. Standard textbook exercises study partial quotients of continued fraction expansions of θ for small n (e.g., [2; p. 388, Probs. 4-7] or [3; p. 204, Probs. 1 and 2]). Note that older notations, e.g., [3], sometimes differ from modern ones by starting the continued fraction with a_1 instead of a_0

Remark 3: Let $C_k = p_k / q_k$ denote the k^{th} convergent of $(\theta - a_0)^{-1}$ for $k = 0, 1, 2, \ldots$. In particular, (1) implies $p_{n-1} / q_{n-1} = [a_1, a_2, \ldots, a_2, a_1, 2a_0]$. The following facts, used in the sequel, are well known (see [1; CF4 and CF1] and [2; p. 385, Eq. 10.17]).

$$p_{n-1} = (2a_0)p_{n-2} + q_{n-2}; \quad q_{n-1} = (2a_0)q_{n-2} + q_{n-3}. \tag{2}$$

$$1 / M = [a_1, a_2, \ldots, a_2, a_1] = p_{n-2} / q_{n-2} > 1; \quad M / N = [a_1, a_2, \ldots, a_2] = q_{n-2} / q_{n-3} > 1. \tag{3}$$

A real

$$x > 1 \tag{4}$$

satisfies the quadratic equation

$$q_{n-1}x^2 + \{q_{n-2} - p_{n-1}\}x - p_{n-2} = 0 \tag{5}$$

if and only if

$$x = (\theta - a_0)^{-1}. \tag{6}$$

Proof of the Theorem: Substitution, using (2), transforms the theorem assertion into the following equivalent claim, which we will prove:

$$\theta=\sqrt{a_0^2+2\frac{q_{n-2}}{p_{n-2}}a_0+\frac{q_{n-3}}{p_{n-2}}}.$$

Let

$$x=\frac{1}{\sqrt{a_0^2+2\frac{q_{n-2}}{p_{n-2}}a_0+\frac{q_{n-3}}{p_{n-2}}}-a_0}.$$

Then (3) implies (4) and straightforward expansion using (2) demonstrates (5). Equations (4) and (5) imply (6) and the result immediately follows.

References:

1. Attila Petho. "Simple Continued Fractions for the Fredholm Numbers." *J. Number Theory* **14** (1982):232-36.
2. Kenneth H. Rosen. *Elementary Number Theory and Its Applications.* Reading, Mass.: Addison Wesley, 1984.
3. James E. Shockley. *Introduction to Number Theory.* New York & Chicago: Holt, Rinehart, and Winston, 1967.

Also solved by the proposer.

Kind of Triggy

H-460 ***Proposed by H.-J. Seiffert, Berlin Germany***
(Vol. 29, no. 4, November 1991)

Define the Fibonacci polynomials by $F_0(x)=0, F_1(x)=1, F_{n+2}(x)=xF_{n+1}(x)+F_n(x)$. Show that, for all positive reals x,.

(a) $$\sum_{k=1}^{n-1}1\Big/\left(x^2+\sin^2\frac{k\pi}{2n}\right)=\frac{(2n-1)F_{2n+1}(2x)+(2n+1)F_{2n-1}(2x)}{4x(x^2+1)F_{2n}(2x)}-\frac{1}{2x^2},$$

(b) $$\sum_{k=1}^{n-1}1\Big/\left(x^2+\sin^2\frac{k\pi}{2n}\right)\sim n/\left(x\sqrt{x^2+1}\right),\text{ as } n\to\infty,$$

(c) $$\sum_{k=1}^{n-1}1\Big/\sin^2\frac{k\pi}{2n}=2(n^2-1)/3.$$

Solution by Paul S. Bruckman, Edmonds, WA

The auxiliary equation for $F_n(2x)$ is given by

$$z^2-2xz-1=0, \tag{1}$$

whose roots r and s are given by

$$r=x+y,\ s=x-y,\ \text{where } y=(x^2+1)^{1/2}. \tag{2}$$

If we set

$$x>\sinh\theta,\ \theta=0, \tag{3}$$

we obtain

$$r = e^{\theta},\ s = -e^{-\theta}, \tag{4}$$

and

$$y = \cosh\theta. \tag{5}$$

Moreover, $F_n(2x) = (r^n - s^n)/(r-s)$, from which we obtain

$$F_{2n}(2x) = \sinh 2n\theta / \cosh\theta; \tag{6}$$

$$F_{2n+1}(2x) = \cosh(2n+1)\theta / \cosh\theta. \tag{7}$$

We may also easily verify the following identities:

$$F_{2n+1}(2x) + F_{2n-1}(2x) = 2\cosh 2n\theta; \tag{8}$$

$$F_{2n+1}(2x) - F_{2n-1}(2x) = 2xF_{2n}(2x). \tag{9}$$

From the recurrence relation defining the F_n's, it readily follows that the leading term of $F_{2n}(2x)$ is $(2x)^{2n-1}$. Moreover, we see from (6) that $F_{2n}(2x) = 0$ if and only if $2n\theta = ki\pi$, $k = 0, \pm 1, \pm 2, \ldots, \pm(n-1)$, or, equivalently, $2n\theta = \pm ki\pi, k = 0, 1, \ldots, n-1$. Thus, $F_{2n}(2x) = 0$ if and only if $x = \sinh\theta = \sinh(\pm ki\pi/2n) = \pm i\sin(k\pi/2n)$. From this, we obtain the factorization:

$$F_{2n}(2x) = 2^{2n-1}x\prod_{k=1}^{n-1}(x^2 + \sin^2 k\pi/2n). \tag{10}$$

Taking the logarithm and derivative in (10), we obtain

$$\frac{F'_{2n}(2x)}{xF_{2n}(2x)} - \frac{1}{2x^2} = \sum_{k=1}^{n-1}(x^2 + \sin^2 k\pi/2n)^{-1} \equiv S_n(x), \text{ say.} \tag{11}$$

Here and in the sequel, the prime symbol denotes differentiation with respect to x.

On the other hand, we may differentiate in (6), using the useful results:

$$y' = x/y = \coth\theta; \tag{12}$$

$$\theta' = 1/y = \operatorname{sech}\theta. \tag{13}$$

Then

$$\begin{aligned} 2F'_{2n}(2x) &= \operatorname{sech}^2\theta[\cosh\theta \cdot 2n\cosh 2n\theta \operatorname{sech}\theta - \sinh 2n\theta \cdot \sinh\theta \operatorname{sech}\theta] \\ &= (x^2+1)^{-1}[2n\cosh 2n\theta - \tanh\theta \sinh 2n\theta] \\ &= (x^2+1)^{-1}[F_{2n+1}(2x) + F_{2n-1}(2x)]n - (x^2+1)^{-1}(x/y)yF_{2n}(2x). \end{aligned}$$

Thus,

$$F'_{2n}(2x)/xF_{2n}(2x) - 1/2x^2 = \frac{2n\big(F_{2n+1}(2x) + F_{2n-1}(2x)\big)}{4x(x^2+1)F_{2n}(2x)} - \frac{1}{2(x^2+1)} - \frac{1}{2x^2}$$

$$= \frac{(2n-1)F_{2n+1}(2x) + (2n+1)F_{2n-1}(2x)}{4x(x^2+1)F_{2n}(2x)} + \frac{F_{2n+1}(2x) - F_{2n-1}(2x)}{4x(x^2+1)F_{2n}(2x)} - \frac{1}{2(x^2+1)} - \frac{1}{2x^2};$$

using (9), this simplifies to:

$$\frac{F'_{2n}(2x)}{xF_{2n}(2x)}-\frac{1}{2x^2}=\frac{(2n-1)F_{2n+1}(2x)+(2n+1)F_{2n-1}(2x)}{4x(x^2+1)F_{2n}(2x)}-\frac{1}{2x^2}\equiv T_n(x),\text{ say.} \tag{14}$$

Comparison of (11) and (14) establishes part (a) of the problem, i.e., $S_n(x)=T_n(x)$.

Also, we may express $T_n(x)$ in the following form:

$$T_n(x)=\frac{2n\cosh 2n\theta-\tanh\theta\sinh 2n\theta}{2(x/y)(x^2+1)\sinh 2n\theta}-\frac{1}{2x^2}$$

$$=\frac{ny\coth 2n\theta}{xy^2}-\frac{1}{2(x^2+1)}-\frac{1}{2x^2}=\frac{n\coth 2n\theta}{xy}-\frac{2x^2+1}{2x^2(x^2+1)},$$

or, since $2x^2+1=2\sinh^2\theta=\cosh 2\theta$ and $4x^2(x^2+1)=4x^2y^2=(2\sinh\theta\cosh\theta)^2=\sinh^2 2\theta$, we obtain

$$S_n(x)=T_n(x)=\frac{n}{x\sqrt{x^2+1}}\coth 2n\theta-\frac{2\cosh 2\theta}{\sinh^2 2\theta}. \tag{15}$$

Now $\lim_{n\to\infty}\coth 2n\theta=\lim_{n\to\infty}\frac{\exp(4n\theta)+1}{\exp(4n\theta)-1}=1.$ Therefore, it follows from (15) that

$$S_n(x)\sim\frac{n}{x(x^2+1)^{1/2}}\text{ as }n\to\infty, \tag{16}$$

which is part (b) of the problem.

We see from (15) that

$$S_n(x)=2n\coth 2n\theta\cdot\operatorname{csch}2\theta-2\cosh 2\theta\cdot\operatorname{csch}^2 2\theta=U_n(\theta),\text{ say.} \tag{17}$$

From the definition in (3), it follows that $\lim_{x\to 0}S_n(x)=S_n(0)=U_n(0)=\lim_{\theta\to\infty}U_n(\theta)$, provided that either limit exists. Also, it appears easier to evaluate this limit by expansion, rather than attempt to apply L'Hôpital's Rule. Toward this end, we require the following expansions:

$$\coth z=z^{-1}(1+z^2/3+0(z^4));\quad \operatorname{csch}z=z^{-1}(1-z^2/6+0(z^4));$$
$$\cosh z=1+\tfrac{1}{2}z^2+0(z^4);\quad \operatorname{csch}^2 z=z^{-2}(1-z^2/3+0(z^4)).$$

Here, "big-O" functions are defined as $z\to 0$. Then

$$U_n(\theta)=\frac{2n}{2n\theta}(1+4n^2\theta^2/3+\cdots)(1/2\theta)(1-2\theta^2/3+\cdots)-2(1+2\theta^2+\cdots)(1/4\theta^2)(1-4\theta^2/3+\cdots),$$

where the "..." notation refers to terms that are $0(\theta^4)$. Then

$$U_n(\theta)=\frac{1}{2\theta^2}\left[1+\frac{2}{3}(2n^2-1)\theta^2-1-\frac{2}{3}\theta^2\right]+0(\theta^2)$$

or

$$U_n(\theta)=\frac{2}{3}(n^2-1)+0(\theta^2). \tag{18}$$

It follows from (18) that $U_n(0)=S_n(0)$ (the limit exists) $=\tfrac{2}{3}(n^2-1)$, which is part (c).

Also solved by the proposer.

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