ON CONSECUTIVE NIVEN NUMBERS

Curtis Cooper and Robert E. Kennedy
Department of Mathematics, Central Missouri State University, Warrensburg, MO 64093
(Submitted June 1991)

INTRODUCTION

In [1] the concept of a Niven number was introduced with the following definition.

Definition: A positive integer is called a Niven number if it is divisible by its digital sum.

Various articles have appeared concerning digital sums and properties of the set of Niven numbers. In particular, it was shown in [2] that no more than 21 consecutive Niven numbers is possible. Here, we will show, in fact, that no more than 20 consecutive Niven numbers is possible and give an infinite number of examples of such sequences.

DIGITAL SUMS AND CARRIES

In what follows, $s(n)$ will denote the digital sum of the positive integer n. The formula

$$s(n) = n - 9\sum_{t\geq 1}\left[\frac{n}{10^t}\right],$$

where the square brackets represent the greatest integer function, is well known and easily derived. Note that the sum has only a finite number of terms since $\left[\frac{n}{10^t}\right] = 0$ where $t > [\log n]$

For integers m and n, we let $c(m+n)$ denote the sum of the "carries" which occur when calculating the sum $m+n$. The following Lemma gives the relationship between $s(m+n)$ and $c(m+n)$.

Lemma: Let m, n be positive integers. Then

$$s(m+n) = s(m) + s(n) - 9c(m+n).$$

Proof: Since

$$s(m) = m - 9\sum_{t\geq 1}\left[\frac{m}{10^t}\right] \quad \text{and} \quad s(n) = n - 9\sum_{t\geq 1}\left[\frac{n}{10^t}\right],$$

it follows that

$$\begin{aligned} s(m) + s(n) &= m + n - 9\sum_{t\geq 1}\left(\left[\frac{m}{10^t}\right] + \left[\frac{n}{10^t}\right]\right) \\ &= s(m+n) + 9\sum_{t\geq 1}\left(\left[\frac{m+n}{10^t}\right] - \left[\frac{m}{10^t}\right] - \left[\frac{n}{10^t}\right]\right). \end{aligned}$$

Noting that the expression

$$\left[\frac{m+n}{10^t}\right] - \left[\frac{m}{10^t}\right] - \left[\frac{n}{10^t}\right]$$

is the carry that occurs when the $(t-1)^{\text{st}}$ right-most digit of n are added, the equality $s(m+n) = s(m) + s(n) - 9c(m+n)$ follows.

In passing, the resder might be interested in proving that $s(mn) = s(m)s(n) - 9c(mn)$ where $c(mn)$ is the sum of the carries that occur in calculating the product of m and n by the usual multiplication algorithm. Here, however, we are concerned with sequences of consecutive Niven numbers.

CONSECUTIVE NIVEN NUMBERS

To discuss consecutive Niven numbers, we will introduce the idea of a decade and a century of numbers. A decade is a set of numbers

$$\{10n, 10n+1, \ldots, 10n+9\}$$

for any nonnegative integer n and a century is a set of numbers

$$\{100n, 100n+1, \ldots, 100n+99\}$$

for any nonnegative integer n. We first observe that in a given decade, either all the odd numbers have an even digital sum or all the odd numbers have an odd digital sum. To make the next observation, let E denote the statement "odd numbers which have an even digital sum" and O denote the statement "odd numbers which have an odd digital sum." We then note that the ten decades in a century alternate either O, E, O, E, O, E, O, E, O, E or E, O, E, O, E, O, E, O, E, O. Finally, we remark that in an E decade, none of the odd numbers can be Niven since their digital sum is even. Thus, the only way to get more than 11 consecutive Niven numbers is to cross a century boundary where the decades between centuries would be

$$\ldots, \text{E}, \text{O}, \text{E}, \text{O} \mid \text{O}, \text{E}, \text{O}, \text{E}, \ldots .$$

Hence, we cannot have more than 21 consecutive Niven numbers and if a list of 21 consecutive Niven numbers exists, it would have to commence with an even Niven number of the form

$$10^t n + 9_{t-1}0$$

where d_r denotes the concatenation of r d's in the decimal representation of an integer. For example,

$$89_5(24)_2 0_3 7 = 89999924240007.$$

Note that d does not have to be a digit. This notation will facilitate an efficient representation for certain large integers.

It is not difficult to find sequences of consecutive Niven numbers. For example, the sequence 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 is an example of 10 consecutive Niven numbers. It is, of course, the smallest such sequence. Other sequences of 10 consecutive Niven numbers can be found, but if a sequence of 21 consecutive Niven numbers could be found, we would have an example of every possible sequence of k consecutive Niven numbers for k = 1, 2, 3, ..., 21. As suggested in the introduction, however, it will be shown that k cannot be larger than 20, and an infinite number of examples with k = 20 will be given. Determining an example with k = 20 involves working with large integers, solving systems of linear congruences, choosing integers with "good" digital sums, a lot of adjusting partial results, and a lot of luck and intuition. Without the use of a computer capable of manipulating large numbers, we could not have found the following sequence in a reasonable length of time.

Let

$$\begin{aligned} a = {} & 4090669070187777592348077471447408839621564801 \\ & 2007115516094806249015486761744582584646124234 \\ & 1540855543641742325745294115007591954820126570 \\ & 087071005523266064292043054902370439430_{1120} \end{aligned}$$

and

$$\begin{aligned} b = {} & 2846362190166818294716429619770154544233311863 \\ & 4187301827478422658543387589306681088151446703 \\ & 2759507916140833155837906335537198825206802774 \\ & 843028314975502097292745955936059236215 69_{1119}0. \end{aligned}$$

Then a has 1296 digits, b has 1298 digits, $s(a) = 720$, and $s(b) = 10870$. Also note that each of

$$\begin{array}{c} 2464645030 \\ 2464645031 \\ \vdots \\ 2464645039 \\ 2464634960 \\ 2464634961 \\ \vdots \\ 2464634969 \end{array}$$

is a factor of a, and

$$\begin{array}{l} 2464645030 \text{ divides } b \\ 2464645031 \text{ divides } b+1 \\ \qquad\qquad\quad \vdots \\ 2464645039 \text{ divides } b+9 \\ 2464634960 \text{ divides } b+10 \\ 2464634961 \text{ divides } b+11 \\ \qquad\qquad\quad \vdots \\ 2464634969 \text{ divides } b+19. \end{array}$$

Now let m be any nonnegative integer and consider

$$x = a_{3423103}0_m b.$$

Then x has $44363342786 + m$ digits, and is a Niven number with $s(x) = 2464645030$. Furthermore, by construction, each of $x+1, x+2, x+3, \ldots, x+19$ is also a Niven number, and a sequence of 20 consecutive Niven numbers has been constructed. Also, since m is an arbitrary nonnegative integer, we have demonstrated an infinite number of such sequences. However, that the methods used in finding such a sequence cannot be used to find 21 consecutive Niven numbers, is revealed by the following discussion.

Suppose that there exists a sequence $x, x+1, x+2, \ldots, x+19, x+20$ of Niven numbers. Then $x \equiv 9_{t-1}0 \pmod{10^t}$ where we may assume that the $(t+1)^{\text{st}}$ right-most digit of x is not a 9. Thus,

(1)	$x \equiv 0 \pmod{s(x)}$	(5)	$x \equiv -4 \pmod{s(x)+4}$
(2)	$x \equiv -1 \pmod{s(x)+1}$	(6)	$x \equiv -5 \pmod{s(x)+5}$
(3)	$x \equiv -2 \pmod{s(x)+2}$	(7)	$x \equiv -6 \pmod{s(x)+6}$
(4)	$x \equiv -3 \pmod{s(x)+3}$	(8)	$x \equiv -7 \pmod{s(x)+7}$

(9)	$x \equiv -8 \pmod{s(x)+8}$	(16)	$x \equiv -15 \pmod{s(x)+15-9t}$
(10)	$x \equiv -9 \pmod{s(x)+9}$	(17)	$x \equiv -16 \pmod{s(x)+16-9t}$
(11)	$x \equiv -10 \pmod{s(x)+10-9t}$	(18)	$x \equiv -17 \pmod{s(x)+17-9t}$
(12)	$x \equiv -11 \pmod{s(x)+11-9t}$	(19)	$x \equiv -18 \pmod{s(x)+18-9t}$
(13)	$x \equiv -12 \pmod{s(x)+12-9t}$	(20)	$x \equiv -19 \pmod{s(x)+19-9t}$
(14)	$x \equiv -13 \pmod{s(x)+13-9t}$	(21)	$x \equiv -20 \pmod{s(x)+11-9t}$
(15)	$x \equiv -14 \pmod{s(x)+14-9t}$		

It should be pointed out that the form of the moduli in the above list follow by the Lemma. That is,

$$\begin{aligned} s(x+k) &= s(x)+s(k)-9c(x+k) \\ &= s(x)+s(k)-9(t-1). \end{aligned}$$

For example,

$$s(x+19) = s(x)+10-9(t-1) = s(x)+19-9t$$

and so the congruence

$$x+19 \equiv 0 \pmod{s(x+19)}$$

may be written as

$$x \equiv -19 \pmod{s(x)+19-9t}.$$

Since

$$x \equiv -20 \pmod{s(x)+11-9t},$$

and

$$x \equiv -11 \pmod{s(x)+11-9t},$$

we immediately have that

$$9 \equiv 0 \pmod{s(x)+11-9t},$$

and so, $s(x)+11-9t = 1, 3,$ or 9. Thus, $9t-s(x) = 2, 8,$ or 10. However, since

$$x \equiv 9_{t-1}0 \pmod{10^t},$$

we see that $s(x) \geq 9t-9$, and it follows that $9t-s(x) = 2$ or 8.

Suppose that $9t-s(x) = 8$. Then, by congruences (11), (12), (14), (16), and (20), we have the system

$$\begin{aligned} x &\equiv 0 \pmod 2 \\ x &\equiv 1 \pmod 3 \\ x &\equiv 2 \pmod 5 \\ x &\equiv 6 \pmod 7 \\ x &\equiv 3 \pmod{11} \end{aligned}$$

which, by the Chinese Remainder Theorem, has the solution

$$x \equiv 6922 \pmod{2310}.$$

But, since $x \equiv 9_{t-1}0 \pmod{10^t}$, it follows that 5 is a factor of x. This cannot be the case if $x \equiv 6922 \pmod{2310}$. Hence, we must conclude that $9t - s(x) \neq 8$.

Now suppose that $9t - s(x) = 2$. Then the congruences (1) through (21) may be rewritten as:

(1)	$x \equiv 0 \pmod{9t-2}$	(12)	$x \equiv 7 \pmod{9}$
(2)	$x \equiv -1 \pmod{9t-1}$	(13)	$x \equiv 8 \pmod{10}$
(3)	$x \equiv -2 \pmod{9t}$	(14)	$x \equiv 9 \pmod{11}$
(4)	$x \equiv -3 \pmod{9t+1}$	(15)	$x \equiv 10 \pmod{12}$
(5)	$x \equiv -4 \pmod{9t+2}$	(16)	$x \equiv 11 \pmod{13}$
(6)	$x \equiv -5 \pmod{9t+3}$	(17)	$x \equiv 12 \pmod{14}$
(7)	$x \equiv -6 \pmod{9t+4}$	(18)	$x \equiv 13 \pmod{15}$
(8)	$x \equiv -7 \pmod{9t+5}$	(19)	$x \equiv 14 \pmod{16}$
(9)	$x \equiv -8 \pmod{9t+6}$	(20)	$x \equiv 15 \pmod{17}$
(10)	$x \equiv -9 \pmod{9t+7}$	(21)	$x \equiv 7 \pmod{9}$,
(11)	$x \equiv 6 \pmod{8}$		

respectively.

Recall that if the system

$$x \equiv r \pmod{m}$$
$$x \equiv s \pmod{n}$$

has a solution, then $\gcd(m, n)$ is a factor of $r - s$. See, for example, [3, Th. 5-11]. Thus, by use of the pairings

(4) with (13)
(6) with (13)
(7) with (13)
(10) with (13),

we have that

$$\gcd(10, 9t+1) = 1$$
$$\gcd(10, 9t+3) = 1$$
$$\gcd(15, 9t+4) = 1$$
$$\gcd(10, 9t+7) = 1,$$

respectively. The fact that x is even together with congruence (2), imply that t is even. But

$$t \equiv 2 \pmod{10} \text{ implies that } \gcd(10, 9t+7) \neq 1,$$
$$t \equiv 4 \pmod{10} \text{ implies that } \gcd(15, 9t+4) \neq 1,$$
$$t \equiv 6 \pmod{10} \text{ implies that } \gcd(10, 9t+1) \neq 1,$$
$$t \equiv 8 \pmod{10} \text{ implies that } \gcd(10, 9t+3) \neq 1,$$

which contradict the above. So, it follows that

$$t \not\equiv 2 \pmod{10}$$
$$t \not\equiv 4 \pmod{10}$$
$$t \not\equiv 6 \pmod{10}$$
$$t \not\equiv 8 \pmod{10}.$$

In addition, the pair of congruences $x \equiv 9_{t-1}0 \pmod{10^t}$ and $x \equiv -7 \pmod{9t+5}$ imply that $\gcd(10^t, 9t+5)$ divides $9_{t-1}7$, from which it follows that 5 cannot be a factor of $\gcd(10^t, 9t+5)$

and so we have that $t \not\equiv 0 \pmod{10}$. Hence, by assuming that $9t - s(x) = 2$, the fact that t is even is contradicted, and we conclude that $9t - s(x) \neq 2$. So, the only two possibilities for $9t - s(x)$ (by assuming that a sequence of 21 consecutive Niven numbers exists) are eliminated. We have, then, the following theorem.

Theorem: There does not exist a sequence of 21 consecutive Niven numbers.

CONCLUSION

Finally, we must admit that we do not know whether or not the sequence of 20 consecutive Niven numbers given here, with $m = 0$, is the smallest such sequence. That is, whether or not the integer $a_{3423103}b$ is the smallest integer that a sequence of 20 consecutive Niven numbers can commence. During the construction of this integer, many alternate possibilities presented themselves, and as mentioned, much intuition and luck were involved. We would, therefore, like to challenge the reader to find the least integer that is the first term in a sequence of 20 consecutive Niven numbers.

REFERENCES

1. R. Kennedy, T. Goodman, & C. Best. "Mathematical Discovery and Niven Numbers." *The MATYC Journal* **14** (1980):21-25.
2. R. Kennedy. "Digital Sums, Niven Numbers, and Natural Density." *Crux Mathematicorum* **8** (1982):129-33.
3. H. Griffin. *Elementary Theory of Numbers.* New York: McGraw-Hill, 1954.

AMS Classification number: 11A63

❖❖❖

Author and Title Index

The AUTHOR, TITLE, KEY-WORD, ELEMENTARY PROBLEMS and ADVANCED PROBLEMS indices for the first 30 volumes of *The Fibonacci Quarterly* have been completed by Dr. Charles K. Cook. Publication of the completed indices is on a 3.5-inch high density disk. The price for a copyrighted version of the disk will be $40.00 plus postage for non-subscribers while subscribers to *The Fibonacci Quarterly* will only need to pay $20.00 plus postage. For additional information or to order a disk copy of the indices, write to:

PROFESSOR CHARLES K. COOK
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SOUTH CAROLINA AT SUMTER
1 LOUISE CIRCLE
SUMTER, SC 29150

The indices have been compiled using WORDPERFECT. Should you wish to order a copy of the indices for another wordprocessor or for a non-compatible IBM machine please explain your situation to Dr. Cook when you place your order and he will try to accommodate you. DO NOT SEND YOUR PAYMENT WITH YOUR ORDER. You will be billed for the indices and postage by Dr. Cook when he sends you the disk. A star is used in the indices to indicate unsolved problems. Furthermore, Dr. Cook is working on a SUBJECT index and will also be classifying all articles by use of the AMS Classification Scheme. Those who purchase the indices will be given one free update of all indices when the SUBJECT index and the AMS classification of all articles published in *The Fibonacci Quarterly* are completed.