MIXED FERMAT CONVOLUTIONS

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1. INTRODUCTION

The k^{th} convolution sequences for Fermat polynomials of the first kind $\left(a_{n,m}^{(k)}(x)\right)$ and the second kind $\left(b_{n,m}^{(k)}(x)\right)$ are defined in this paper. Generating functions, recurrence relations, and explicit representations are given for these polynomials. A differential equation that corresponds to polynomials of type $\left(a_{n,m}^{(k)}(x)\right)$ is presented. Finally, k^{th} convolutions of mixed Fermat polyno-mials of $\left(c_{n,m}^{(s,r)}(x)\right)$ are defined. In some special cases, polynomials of $\left(c_{n,m}^{(s,r)}(x)\right)$ are transformed into already known polynomials of $\left(a_{n,m}^{(k)}(x)\right)$ and of $\left(b_{n,m}^{(k)}(x)\right)$.

2. POLYNOMIALS $a_{n,m}^{(k)}(x)$

A. F. Horadam [2] defined Fermat polynomials of the first kind $A_n(x)$ and the second kind $B_n(x)$ by

(2.1) $$A_n(x) = xA_{n-1}(x) - 2A_{n-2}(x),\ A_{-1}(x) = 0,\ A_0(x) = 1,$$

and

(2.2) $$B_n(x) = xB_{n-1}(x) - 2b_{n-2}(x),\ B_0(x) = 2,\ B_1(x) = x.$$

Their generating functions are

(2.3) $$(1 - xt + 2t^2)^{-1} = \sum_{n=0}^{\infty} A_n(x)t^n$$

and

(2.4) $$\frac{1-2t^2}{1-xt+2t^2} = \sum_{n=0}^{\infty} B_n(x)t^n.$$

From (2.1) and (2.2), we can find a few members of the sequence of polynomials $A_n(x)$ and $B_n(x)$:

$$A_1(x) = x,\ A_2(x) = x^2 - 2,\ A_3(x) = x^3 - 4x,\ A_4(x) = x^4 - 6x^2 + 4,$$

and

$$B_2(x) = x^2 - 4,\ B_3(x) = x^3 - 6x,\ B_4(x) = x^4 - 8x^2 + 8.$$

H. W. Gould [1] studied a class of generalized Humbert polynomials $P_n(m, x, y, p, C)$ defined by

$$(C - mxt + yt^m)^p = \sum_{n=0}^{\infty} P_n(m, x, y, p, C)t^n,$$

where $m \geq 1$ is integer and the other parameters are unrestricted in general. The recurrence relation for the generalized Humbert polynomials is

$$CnP_n - m(n-1-p)xP_{n-1} + (n-m-mp)yP_{n-m} = 0,\ n \ge m \ge 1,$$

where we put $P_n = P_n(m, x, y, p, C)$.

In this paper we consider the polynomials $\left(a_{n,m}^{(k)}(x)\right)$ defined by

$$a_{n,m}^{(k)}(x) = P_n(m, x/m, 2, -(k+1), 1).$$

Their generating function is given by

(2.5) $$F(x,t) = (1 - xt + 2t^m)^{-(k+1)} = \sum_{n=0}^{\infty} a_{n,m}^{(k)}(x)t^n.$$

Comparing (2.3) to (2.5), we can conclude that

$$a_{n,2}^{(0)}(x) = A_n(x) \quad \text{[Fermat polynomials (2.1)]}.$$

Development of the function (2.5) gives

$$\sum_{n=0}^{\infty} a_{n,m}^{(k)}(x)t^n = \sum_{n=0}^{\infty} \frac{(k+1)_n}{n!} t^n (x - 2t^{m-1})^n$$

$$= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{\left[\frac{n}{m}\right]} (-2)^i \frac{(k+1)_{n-(m-1)i}}{i!(n-mi)!} x^{n-mi} \right) t^n.$$

Comparison of coefficients of t^n in the last equation shows that polynomials $\left(a_{n,m}^{(k)}(x)\right)$ possess explicit representation as follows:

(2.6) $$a_{n,m}^{(k)}(x) = \sum_{i=0}^{\left[\frac{n}{m}\right]} (-2)^i \frac{(k+1)_{n-(m-1)i}}{i!(n-mi)!} x^{n-mi}.$$

If we differentiate the function $F(x,t)$ (2.5) with respect to t, and compare coefficients of t^n, we get the three-term recurrence relation

$$na_{n,m}^{(k)}(x) = x(n+k)a_{n-1,m}^{(k)}(x) - 2(n+mk)a_{n-m,m}^{(k)}(x),\ n \ge m.$$

The initial starting polynomials are

$$a_{0,m}^{(k)}(x) = 0,\quad a_{n,m}^{(k)}(x) = \frac{(k+1)_n}{n!} x^n,\quad n = 1, 2, \ldots, m-1.$$

Then, if we differentiate the polynomials $a_{n,m}^{(k)}(x)$ (2.6) s times, term by term, we get the equality [1]:

$$D^s a_{n,m}^{(k)}(x) = (k+1)_s a_{n-s,m}^{(k+s)}(x),\ n \ge s.$$

Let the sequence $(f_r)_{n=0}^{n}$ be given by $f_r = f(r)$, where

$$f(t) = (n-t)\left(\frac{n-t+m(k+1+t)}{m}\right)_{m-1}.$$

Let Δ be the standard difference operator defined by $\Delta f_r = f_{r+1} - f_r$, and its power by

$$\Delta^0 f_r = f_r,\quad \Delta^k f_r = \Delta\left(\Delta^{k-1} f_r\right).$$

We find that the next property of $a_{n,m}^{(k)}(x)$ is very interesting.

The polynomial $a_{n,m}^{(k)}(x)$ is a particular solution of the linear homogeneous differential equation of the m^{th} order [4],

$$y^{(m)} + \sum_{s=0}^{m} a_s x^s y^{(s)} = 0, \tag{2.7}$$

with coefficients a_s $(s = 0, 1, \dots, m)$ given by

$$a_s = \frac{1}{2ms!}\Delta^s f_0. \tag{2.8}$$

From (2.8), we get

$$a_0 = \frac{1}{2m} n\left(\frac{n+m(k+1)}{m}\right)_{m-1}$$

$$a_1 = \frac{1}{2m}\left((n-1)\left(\frac{n-1+m(k+2)}{m}\right)_{m-1} - n\left(\frac{n+m(k+1)}{m}\right)_{m-1}\right).$$

Since

$$f(t) = -\left(\frac{m-1}{m}\right)^{m-1} t^m + \text{ term of lower degree,}$$

we see that

$$a_m = -\frac{1}{2m}\left(\frac{m-1}{m}\right)^{m-1}.$$

For $m = 2$, the differential equation (2.7) is

$$\left(1 - \frac{1}{8}x^2\right)y'' - \frac{2k=3}{8}xy' + \frac{n}{8}(n+2k+2)y = 0,$$

and it corresponds to the polynomials $a_{n,2}^{(k)}(x)$.

For $m = 2$ and $k = 0$, the equation (2.7) is

$$\left(1 - \frac{1}{8}x^2\right)y'' - \frac{3}{8}xy' + \frac{n}{8}(n+2)y = 0,$$

and it corresponds to Fermat polynomials of the first kind $A_n(x)$.

3. POLYNOMIALS $b_{n,m}^{(k)}(x)$

In this section we introduce a class of polynomials $\left(b_{n,m}^{(k)}(x)\right)$, $k \in N$.

Definition 3.1: The polynomials $b_{n,m}^{(k)}(x)$ are defined by

$$F(x,t)=\left(\frac{1-2t^m}{1-xt+2t^m}\right)^{k+1}=\sum_{n=0}^{\infty}b_{n,m}^{(k)}(x)t^n. \tag{3.1}$$

Comparing (2.4) to (3.1), we can see that

$$b_{n,2}^{(0)}(x)=B_n(x)\quad\text{[Fermat polynomials (2.2)]}.$$

Expanding the left-hand side of (3.1), we obtain the explicit formula

$$b_{n,m}^{(k)}(x)=\sum_{i=0}^{k+1}(-2)^i\binom{k+1}{i}a_{n-mi,m}^{(k)}(x). \tag{3.2}$$

For $m=2$ and $k=0$, the formula (3.2) is

$$b_{n,2}^{(0)}(x)=a_{n,2}^{(0)}(x)-2a_{n-2,2}^{(0)}(x).$$

That is,

$$B_n(x)=A_n(x)-2A_{n-2}(x).$$

and it corresponds to the known relation between the Fermat polynomials $A_n(x)$ and $B_n(x)$.

4. MIXED FERMAT CONVOLUTIONS

A. F. Horadam and J. M. Mahon [3] studied a class of polynomials $(\pi_n^{(a,b)}(x))$, mixed Pell polynomials. Similarly, we define and then carefully study polynomials $(c_{n,m}^{(s,r)}(x))$, mixed Fermat convolutions, where all parameters are natural numbers.

Definition 4.1: The polynomials $(c_{n,m}^{(s,r)}(x))$ are given by

$$F(x,t)=\frac{(1-2t^m)^r}{(1-xt+2t^m)^{r+s}}=\sum_{n=0}^{\infty}c_{n,m}^{(s,r)}(x)t^n, \tag{4.1}$$

on condition that $s+r\geq 1$.

The polynomials $(c_{n,m}^{(s,r)}(x))$ have some interesting characteristics, some of which are described in the results that follow.

Theorem 4.1: The polynomials $(c_{n,m}^{(s,r)}(x))$ have the representation

$$c_{n,m}^{(s,r)}(x)=\sum_{i=0}^{r-j}(-2)^i\binom{r-j}{i}c_{n-mi,m}^{(r+s-j,j)}(x). \tag{4.2}$$

Proof: By using (4.1), we obtain

$$\sum_{n=0}^{\infty}c_{n,m}^{(s,r)}(x)t^n=(1-2t^m)^{r-j}\cdot\frac{1}{(1-xt+2t^m)^{r+s-j}}\cdot\left(\frac{1-2t^m}{1-xt+2t^m}\right)^j$$

$$=\sum_{n=0}^{\infty}\sum_{i=0}^{r-j}(-2)^i\binom{r-j}{i}c_{n-mi,m}^{(r+s-j,j)}(x)t^n.$$

If we compare coefficients of t^n in the last equality, we have (4.2). Using (4.1) again, we obtain the following representation:

$$c_{n,m}^{(s,r)}(x) = \sum_{k=0}^{\infty} a_{n-k,m}^{(s-1)}(x) b_{k,m}^{(r-1)}(x).$$

Also, we see that

$$F(x,t) = \frac{(1-2t^m)^r}{(1-xt+2t^m)^{r+s}} = (1-2t^m)^r \sum_{n=0}^{\infty} a_{n,m}^{(r+s-1)}(x)t^m = \sum_{n=0}^{\infty}\left(\sum_{i=0}^{r}(-2)^i \binom{r}{i} a_{n-mi,m}^{(r+s-1)}(x)\right)t^n.$$

From the last equality, we can conclude that

$$c_{n,m}^{(s,r)}(x) = \sum_{i=0}^{r}(-2)^i \binom{r}{i} a_{n-mi,m}^{(r+s-1)}(x).$$

The Fermat polynomials of the first and of the second kind satisfy a three-term recurrence relation. But, mixed Fermat convolutions satisfy a four-term recurrence relation of unstandard form, which we prove in the following result.

Theorem 4.2: The polynomials $c_{n,m}^{(s,r)}(x)$ satisfy the recurrence relation

$$nc_{n,m}^{(s,r)}(x) = -2mrc_{n-m,m}^{(s+1,r-1)}(x) + x(r+s)c_{n-1,m}^{(s+1,r)}(x) - 2m(r+s)c_{n-m,m}^{(s+1,r)}(x),\ n \ge m. \tag{4.3}$$

Proof: If we differentiate $F(x,t)$, (4.1), with respect to t, we get

$$\sum_{n=1}^{\infty} nc_{n,m}^{(s,r)}(x)t^{n-1} = -2mrt^{m-1}\sum_{n=0}^{\infty} c_{n,m}^{(s+1,r-1)}(x)t^n + (r+s)(x-2mt^{m-1})\sum_{n=0}^{\infty} c_{n,m}^{(s+1,r)}(x)t^n.$$

Comparing coefficients of t^n in the last equality, we have (4.3).

If we differentiate $F(x,t)$, (4.1), with respect to x, k times, term by term, we find that the polynomials $c_{n,m}^{(s,r)}(x)$ satisfy the equality

$$D^k c_{n,m}^{(s,r)}(x) = (r+s)_k c_{n-k,m}^{(s+k,r)}(x) \quad (n \ge k). \tag{4.4}$$

Special Cases

Starting with the equality

$$\frac{(1-2t^m)^{r+s}}{(1-xt+2t^m)^{2r+2s}} = \frac{(1-2t^m)^r}{(1-xt+2t^m)^{r+s}} \cdot \frac{(1-2t^m)^s}{(1-xt+2t^m)^{s+r}},$$

we get

$$\sum_{n=0}^{\infty} c_{n,m}^{(s+r,s+r)}(x)t^n = \left(\sum_{n=0}^{\infty} c_{n,m}^{(s,r)}(x)t^n\right)\left(\sum_{n=0}^{\infty} c_{n,m}^{(r,s)}(x)t^n\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} c_{n-k,m}^{(s,r)}(x)c_{k,m}^{(r,s)}(x)\right)t^n.$$

From the last equality, we obtain

$$c_{n,m}^{(s+r,s+r)}(x) = \sum_{k=0}^{n} c_{n-k,m}^{(s,r)}(x)c_{k,m}^{(r,s)}(x). \tag{4.5}$$

For $r = s$, the equality (4.5) is

$$c_{n,m}^{(2s,2s)}(x) = \sum_{k=0}^{n} c_{n-k,m}^{(s,s)}(x) c_{k,m}^{(s,s)}(x).$$

From the equalities (2.5), (3.1), and (4.1), we obtain:

(4.6) $$c_{n,m}^{(s,0)}(x) = a_{n,m}^{(s-1)}(x), \text{ for } r = 0$$

and

(4.7) $$c_{n,m}^{(0,r)}(x) = b_{n,m}^{(r-1)}(x), \text{ for } s = 0.$$

According to (4.4), (4.6), and (4.7), we get the inequalities

$$D^k a_{n,m}^{(s-1)}(x) = (s)_k a_{n-k,m}^{(s+k-1)}(x), \text{ for } r = 0$$

and

$$D^k b_{n,m}^{(r-1)}(x) = (r)_k c_{n-k,m}^{(k,r)}(x), \text{ for } s = 0.$$

For $r = 0$, the equality (4.5) becomes

$$c_{n,m}^{(s,s)}(x) = \sum_{k=0}^{n} a_{n-k,m}^{(s-1)}(x) b_{k,m}^{(s-1)}(x).$$

According to (4.3) and (4.5), we have

$$n\sum_{k=0}^{\infty} c_{n-k,m}^{(s,0)}(x) c_{k,m}^{(0,s)}(x) = -2msc_{n-m,m}^{(s+1,s-1)}(x) + 2xsc_{n-1,m}^{(s+1,s)}(x) - 4msc_{n-m,m}^{(s+1,s)}(x), \; n \geq m.$$

From the equalities (4.2) and (4.7), for $j = s = 0$, $r = k+1$, it follows that

$$b_{n,m}^{(k)}(x) = \sum_{i=0}^{k+1} (-2)^i \binom{k+1}{i} a_{n-mi,m}^{(k)}(x).$$

Finally, from the equalities (4.2) and (4.6), for $j = r = 0$, $s = k+1$, we see that

$$a_{n,m}^{(k)}(x) = \sum_{i=0}^{k+1} (-2)^i \binom{k+1}{i} a_{n-mi,m}^{(k)}(x).$$

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