# TRIPLE FACTORIZATION OF SOME RIORDAN MATRICES


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## 1. INTRODUCTION

When examining a combinatorial sequence, generating functions are often useful. That is, if we are interested in analyzing the sequence $a_0, a_1, a_2, \ldots$, we investigate the formal power series

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots .$$

In a recent paper [2], techniques are discussed that assist in finding closed-form expressions for the formal power series for a select, but large, set of combinatorial sequences. The methods involve using infinite matrices and the Riordan group. The Riordan group is defined in section 2 of this paper. Each matrix, $L$, in the Riordan group is associated with a combinatorial sequence and with a matrix, $S_L$, called the Stieltjes matrix. $S_L$ is defined in section 3. In this paper, we show that when $S_L$ is tridiagonal, then $L = PCF$, where the first factor $P$ is a Pascal-type matrix, the second factor $C$ involves the generating function for the Catalan numbers, and the third factor $F$ involves the Fibonacci generating function. The following is an example:

$$\begin{bmatrix} 1 & & & & & & & \cdot \\ 3 & 1 & & & & \mathbf{0} & & \cdot \\ 9 & 5 & 1 & & & & & \cdot \\ 27 & 18 & 7 & 1 & & & & \cdot \\ 81 & 56 & 31 & 9 & 1 & & & \cdot \\ 243 & 162 & 109 & 48 & 11 & 1 & & \cdot \\ 729 & 458 & 332 & 194 & 69 & 13 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & & & & & \cdot \\ 1 & 1 & & & & \mathbf{0} & & \cdot \\ 1 & 2 & 1 & & & & & \cdot \\ 1 & 3 & 3 & 1 & & & & \cdot \\ 1 & 4 & 6 & 4 & 1 & & & \cdot \\ 1 & 5 & 10 & 10 & 5 & 1 & & \cdot \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 & & & & & & & \cdot \\ 0 & 1 & & & & \mathbf{0} & & \cdot \\ -1 & 0 & 1 & & & & & \cdot \\ 0 & -2 & 0 & 1 & & & & \cdot \\ 2 & 0 & -3 & 0 & 1 & & & \cdot \\ 0 & 5 & 0 & -4 & 0 & 1 & & \cdot \\ -5 & 0 & 9 & 0 & -5 & 0 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 & & & & & & & \cdot \\ 1 & 1 & & & & \mathbf{0} & & \cdot \\ 2 & 1 & 1 & & & & & \cdot \\ 3 & 2 & 1 & 1 & & & & \cdot \\ 5 & 3 & 2 & 1 & 1 & & & \cdot \\ 8 & 5 & 3 & 2 & 1 & 1 & & \cdot \\ 13 & 8 & 5 & 3 & 2 & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

The matrices in the Riordan group are infinite and lower triangular. So the example shows only the first seven rows. The first factor on the right is the Pascal matrix. The first column in the second factor has $C(-x^2)$ as generating function, where

$$C(x) = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n}x^n$$

is the generating function for the Catalan numbers. The third factor has the Fibonacci numbers in each column. See section 6 for further examples of this triple factorization.


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In section 2, we define the Riordan group $R$ and list some properties that we use in the proofs of the propositions which are given in section 4. In section 3, we discuss the unique Stieltjes matrix $S_L$ associated with each $L$ in the Riordan group. In this paper, we concentrate on the subset of $R$ given by $R_T = \{L \in R : S_L \text{ is tridiagonal}\}$. In section 5, we derive a recurrence relation for the sequence associated with each member of $R_T$, and we discuss the asymptotic behavior of these sequences. In section 6, we provide two examples involving well-known sequences. For each example, we give the triple factorization, the Stieltjes matrix, the recurrence relation and asymptotic behavior of the corresponding sequence.

## 2. THE RIORDAN GROUP

A detailed description of this group is given in [2]. Here we provide a brief summary.

Let $M = (m_{ij})_{i,j\geq 0}$ be an infinite matrix with elements from C, the set of complex numbers. Let $c_i(x)$ be the generating function of the $i^{\text{th}}$ column of $M$. That is

$$c_i(x) = \sum_{n=0}^{\infty} m_{n,i} x^n.$$

We call $M$ a Riordan matrix if $c_i(x) = g(x)[f(x)]^i$, where

$$g(x) = 1 + g_1 x + g_2 x^2 + g_3 x^3 + \cdots, \quad \text{and} \quad f(x) = x + f_2 x^2 + f_3 x^3 + \cdots.$$

In this case, we write $M = (g(x), f(x))$. We denote by $R$ the set of Riordan matrices. $R$ is a group under matrix multiplication with the following properties:

(i) $(g(x), f(x)) * (h(x), \ell(x)) = (g(x)h(f(x)), \ell(f(x)))$.

(ii) $I = (1, x)$ is the identity element.

(iii) The inverse of $M$ is given by

$$M^{-1} = \left(\frac{1}{g(\bar{f}(x))}, \bar{f}(x)\right),$$

where $\bar{f}$ is the compositional inverse of $f$.

(iv) If $(a_0, a_1, a_2, \ldots)^T$ is a column vector with generating function $A(x)$, then multiplying $M = (g(x), f(x))$ on the right by this column vector yields a column vector with generating function $B(x) = g(x)A(f(x))$.

## 3. STIELTJES MATRIX

Let $L$ be Riordan and let $\overline{L}$ be the matrix obtained from $L$ by deleting the first row. For example, if $I$ is the identity, we have

$$\bar{I} = \begin{bmatrix} 0 & 1 & & & \mathbf{0} & . \\ 0 & 0 & 1 & & & . \\ 0 & 0 & 0 & 1 & & . \\ 0 & 0 & 0 & 0 & 1 & . \\ . & . & . & . & . & . \end{bmatrix}.$$

Observe that $\overline{L} = \overline{I}L$. There exists a unique matrix, $S_L$, such that $LS_L = \overline{L}$. We call this matrix the Stieltjes matrix of $L$.

***Example:*** If

$$L = \left(\frac{1}{1-x}, \frac{x}{1-x}\right) = \begin{bmatrix} 1 & & & & \mathbf{0} & \cdot \\ 1 & 1 & & & & \cdot \\ 1 & 2 & 1 & & & \cdot \\ 1 & 3 & 3 & 1 & & \cdot \\ 1 & 4 & 6 & 4 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix},$$

then

$$S_L = \begin{bmatrix} 1 & 1 & & & \mathbf{0} & \cdot \\ 0 & 1 & 1 & & & \cdot \\ 0 & 0 & 1 & 1 & & \cdot \\ 0 & 0 & 0 & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

## 4. PROPOSITIONS

***Proposition 1:*** If $L = (g(x), f(x))$ is Riordan and $S_L$ is tridiagonal, then

***(a)***

$$S_L = \begin{bmatrix} b_0 & 1 & & & & \mathbf{0} & \cdot \\ \lambda_1 & b & 1 & & & & \cdot \\ 0 & \lambda & b & 1 & & & \cdot \\ 0 & 0 & \lambda & b & 1 & & \cdot \\ 0 & 0 & 0 & \lambda & b & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix},$$

***(b)*** $\quad f = x(1 + bf + \lambda f^2)$ and $g = \dfrac{1}{1 - b_0x - \lambda_1 xf}$ iff $S_L$ is as in (a).

***Proof:*** Let

$$S_L = \begin{bmatrix} b_0 & 1 & & & \mathbf{0} & \cdot \\ \lambda_1 & b_1 & 1 & & & \cdot \\ 0 & \lambda_2 & b_2 & 1 & & \cdot \\ 0 & 0 & \lambda_3 & b_3 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

With $c_i(x)$ the generating function for the $i^{\text{th}}$ column of $L$, $i \geq 0$, we have $c_i = gf^i$. By looking at the first column of $LS_L$ and $\overline{L}$, we obtain $b_0xg + \lambda_1 xgf = g - 1$, i.e.,

$$g(x) = \frac{1}{1 - b_0x - \lambda_1 xf}.$$

For $i \geq 1$, we obtain from $LS_L = \overline{L}$,

$$c_i = x(c_{i-1} + b_i c_i + \lambda_{i+1} c_{i+1}).$$
$$\therefore \quad gf^i = x(gf^{i-1} + b_i gf^i + \lambda_{i+1} gf^{i+1})$$
$$\Leftrightarrow \quad f = x(1 + b_i f + \lambda_{i+1} f^2)$$
$$\therefore \quad 0 = (b_i - b_j)f + (\lambda_{i+1} - \lambda_{j+1})f^2 \text{ for all } i \text{ and } j \geq 1.$$

$$\therefore \quad b_i = b_j \text{ and } \lambda_{i+1} = \lambda_{j+1} \text{ for all } i \text{ and } j \geq 1.$$
$$\therefore \quad \text{we can take } b = b_1 = b_2 = b_3 = \cdots$$
$$\text{and } \lambda = \lambda_2 = \lambda_3 = \lambda_4 = \cdots.$$
$$\therefore \quad f = x(1 + bf + \lambda f^2).$$

***Remark:*** If $S_L$ is tridiagonal, it has the form in (a) and then either

***(a)*** $\lambda = 0$ and $f = \dfrac{x}{1-bx}$ and $g = \dfrac{1-bx}{1-(b_1+b_0)x+(bb_0-\lambda_1)x^2}$ or

***(b)*** $\lambda \neq 0$ and $f = \dfrac{1-bx-\sqrt{(b^2-4\lambda)x^2-2bx+1}}{2\lambda x}$ and $g = \dfrac{1}{1-b_0x-\lambda_1 xf}$.

***Proposition 2:*** If $L = (g, f)$ is Riordan, then $S_L = S_{L^*} + bI$ if and only if $L = P^b L^*$, where

$$P^b = \left(\frac{1}{1-bx}, \frac{x}{1-bx}\right) = \begin{bmatrix} 1 & & & & & \cdot \\ b & 1 & & & \mathbf{0} & \cdot \\ b^2 & 2b & 1 & & & \cdot \\ b^3 & 3b^2 & 3b & 1 & & \cdot \\ b^4 & 4b^3 & 6b^2 & 4b & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \text{(cf. [3], p. 171)}$$

***Proof:*** Note that

$$S_{P^b} = \begin{bmatrix} b & 1 & & & & & \cdot \\ 0 & b & 1 & & & \mathbf{0} & \cdot \\ 0 & 0 & b & 1 & & & \cdot \\ 0 & 0 & 0 & b & 1 & & \cdot \\ 0 & 0 & 0 & 0 & b & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} = bI + \bar{I}.$$

So,

$$\begin{aligned} L = P^b L^* &\Rightarrow \bar{I}L = \bar{I}P^b L^* \\ &\Rightarrow \bar{L} = \overline{P^b} L^* = P^b S_{P^b} L^* = P^b(bI + \bar{I})L^* = bP^b L^* + P^b \bar{I} L^* \\ &= bL + P^b \overline{L}^* = bL + P^b L^* S_{L^*} = L(bI + S_{L^*}). \end{aligned}$$

Conversely, suppose $S_L = bI + S_{L^*}$. Then

$$\begin{aligned} P^b L^*(bI + S_{L^*}) &= \underline{bIP^b L^*} = P^b \overline{L}^* = \underline{P^b bIL^*} + P^b \bar{I} L^* = P^b(bI + \bar{I})L^* \\ &= \overline{P^b} L^* = \bar{I}(P^b L^*) = \overline{P^b L^*}. \end{aligned}$$

***Proposition 3:*** If $L=(g,f)$ is Riordan and

$$S_L=\begin{bmatrix} b+\varepsilon & 1 & & & & & . \\ \lambda+\delta & b & 1 & & & & . \\ 0 & \lambda & b & 1 & & & . \\ 0 & 0 & \lambda & b & 1 & & . \\ 0 & 0 & 0 & \lambda & b & 1 & . \\ . & . & . & . & . & . & . \end{bmatrix},$$

then $L=P^bL_1$, where

$$S_{L_1}=\begin{bmatrix} \varepsilon & 1 & & & & . \\ \lambda+\delta & 0 & 1 & & & . \\ 0 & \lambda & 0 & 1 & & . \\ 0 & 0 & \lambda & 0 & 1 & . \\ 0 & 0 & 0 & \lambda & 0 & . \\ . & . & . & . & . & . \end{bmatrix}.$$

***Proof:*** This follows immediately from Proposition 2.

***Proposition 4 (PCF Factorization):*** In Proposition 3, $L_1=C_\lambda F_{\varepsilon,\delta}$, where

$$C_\lambda=\left(c(\lambda x^2),xc(\lambda x^2)\right)$$

with

$$c(x)=1+x[c(x)]^2=\frac{1-\sqrt{1-4x}}{2x}=\sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^n,\text{ and }F_{\varepsilon,\delta}=\left(\frac{1}{1-\varepsilon x-\delta x^2},x\right).$$

***Proof:*** Let $L_1=(g_1,f_1)$. Then, from Proposition 1, we must have, when $\lambda\neq 0$,

$$f_1=\frac{1-\sqrt{1-4\lambda x^2}}{2\lambda x}\qquad\text{and}\qquad g_1=\frac{1}{1-\varepsilon x-(\lambda+\delta)xf_1}.$$

$$f_1=xc(\lambda x^2)\qquad\text{and}\qquad g_1=\frac{1}{1-\varepsilon x-(\lambda+\delta)x^2c(\lambda x^2)}.$$

Now, from section 2, property 1, we have

$$C_\lambda F_{\varepsilon,\delta}=\left(c(\lambda x^2),\,xc(\lambda x^2)\right)*\left(\frac{1}{1-\varepsilon x-\delta x^2},x\right)=\left(\frac{c(\lambda x^2)}{1-\varepsilon xc(\lambda x^2)-\delta[xc(\lambda x^2)]^2},xc(\lambda x^2)\right).$$

But

$$\frac{1}{1-\varepsilon x-(\lambda+\delta)x^2c(\lambda x^2)}=\frac{c(\lambda x^2)}{1-\varepsilon xc(\lambda x^2)-\delta x^2[c(\lambda x^2)]^2}$$
$$\Leftrightarrow 1-\varepsilon xc(\lambda x^2)-\delta x^2[c(\lambda x^2)]^2=c(\lambda x^2)-\varepsilon xc(\lambda x^2)-(\lambda+\delta)x^2[c(\lambda x^2)]^2$$
$$\Leftrightarrow 1-c(x^2)+\lambda x^2[c(\lambda x^2)]^2=0.$$

## 5. RECURRENCE RELATIONS AND ASYMPTOTICS

We have proved that when $L = (f(x), g(x))$ is Riordan and $S_L$ is tridiagonal with the form

$$S_L = \begin{bmatrix} b_0 & 1 & & & & & . \\ \lambda_1 & b & 1 & & & & . \\ 0 & \lambda & b & 1 & & & . \\ 0 & 0 & \lambda & b & 1 & & . \\ 0 & 0 & 0 & \lambda & b & 1 & . \\ . & . & . & . & . & . & . \end{bmatrix}$$

then

$$f(x) = \sum_{n=0}^{\infty} f_n x^n = \frac{1 - bx - \sqrt{(b^2 - 4\lambda)x^2 - 2bx + 1}}{2\lambda x}$$

and

$$g(x) = \sum_{n=0}^{\infty} g_n x^n = \frac{1}{1 - b_0 x - \lambda_1 x f(x)}.$$

Using the J.C. P. Miller formula (see Henrici [3]), we obtain for $f_n$ the three-term recurrence

$$(n+2) f_{n+1} = (2n+1) b f_n + (1-n)(b^2 - 4\lambda) f_{n-1},$$

and for $g_n$ the five-term recurrence

$$\begin{aligned} nAg_n &= [(2n-3)bA - nB]g_{n-1} + [(2n-3)bB + (3-n)(b^2-4\lambda)A - nC]g_{n-2} \\ &\quad + [(2n-3)bC + (3-n)(b^2-4\lambda)B]g_{n-3} + [(3-n)(b^2-4\lambda)C]g_{n-4} \end{aligned}$$

where $A = \lambda - \lambda_1$, $B = \lambda_1 b + \lambda_1 b_0 - 2\lambda b_0$, $C = \lambda_1^2 - \lambda_1 b b_0 + \lambda b_0^2$. For the asymptotics, we use the methods described in Wilf [4, Ch. 5]. For large $n$, we obtain

$$f_n \sim \frac{(n+1)^{-3/2}(b + 2\sqrt{\lambda})^{n+1/2}}{2\lambda^{3/4}\sqrt{\pi}}$$

where $b^2 > 4\lambda > 0$.

Because there are too many cases to consider, we do not attempt to provide a general formula for the asymptotic value of $g_n$. However, the examples in section 6 illustrate the techniques involved.

## 6. EXAMPLES

***Example 1*—Big Schröder Numbers:** If we take $\lambda = 2$, $b = 3$, $\lambda_1 = \lambda + \delta = 2$, and $b_0 = b + \varepsilon = 2$, then

$$f = \frac{1 - 3x - \sqrt{x^2 - 6x + 1}}{4x} \quad \text{and} \quad g = \frac{1 - x - \sqrt{x^2 - 6x + 1}}{2x}.$$

$g$ is the generating function for the Big Schröder numbers [1].

$$L=(g,f) \text{ with } S_L=\begin{bmatrix} 2 & 1 & & & & & . \\ 2 & 3 & 1 & & \mathbf{0} & & . \\ 0 & 2 & 3 & 1 & & & . \\ 0 & 0 & 2 & 3 & 1 & & . \\ 0 & 0 & 0 & 2 & 3 & 1 & . \\ . & . & . & . & . & . & . \end{bmatrix}.$$

and

$$L=P^3C_2F_{-1,0}=\left(\frac{1}{1-3x},\frac{x}{1-3x}\right)*\left(c(2x^2),xc(2x^2)\right)*\left(\frac{1}{1+x},x\right)$$

$$=\begin{bmatrix} 1 & & & & & . \\ 3 & 1 & & \mathbf{0} & & . \\ 9 & 6 & 1 & & & . \\ 27 & 27 & 9 & 1 & & . \\ 81 & 108 & 54 & 12 & 1 & . \\ . & . & . & . & . & . \end{bmatrix}\begin{bmatrix} 1 & & & & & . \\ 0 & 1 & & \mathbf{0} & & . \\ 2 & 0 & 1 & & & . \\ 0 & 4 & 0 & 1 & & . \\ 8 & 0 & 6 & 0 & 1 & . \\ . & . & . & . & . & . \end{bmatrix}\begin{bmatrix} 1 & & & & & . \\ -1 & 1 & & \mathbf{0} & & . \\ 1 & -1 & 1 & & & . \\ -1 & 1 & -1 & 1 & & . \\ 1 & -1 & 1 & -1 & 1 & . \\ . & . & . & . & . & . \end{bmatrix}$$

$$=\begin{bmatrix} 1 & & & & & . \\ 2 & 1 & & \mathbf{0} & & . \\ 6 & 5 & 1 & & & . \\ 22 & 23 & 8 & 1 & & . \\ 90 & 107 & 49 & 11 & 1 & . \\ . & . & . & . & . & . \end{bmatrix}.$$

***Recurrence Relations:*** Here $A=0,\ B=2,\ C=0,\ (n+2)f_{n+1}=3(2n+1)f_n+(1-n)f_{n-1}$ for $n\ge 1$. $f_0=0,\ f_1=1.$ $ng_{n-1}=3(2n-3)g_{n-2}+(3-n)g_{n-3},\ \ (n+1)g_n=3(2n-1)g_{n-1}+(2-n)g_{n-2},$ for $n\ge 2.$ $g_0=1,\ g_1=2.$

***Asymptotics:***

$$f_n=[x^n]f(x)\sim\frac{(n+1)^{-3/2}(b+2\sqrt{\lambda})^{n+1/2}}{2\lambda^{3/4}\sqrt{\pi}}=\frac{(n+1)^{-3/2}(3+2\sqrt{2})^{n+1/2}}{2\cdot 2^{3/4}\cdot\sqrt{\pi}},$$

$$g_n=[x^n]g(x)=[x^n]\frac{1-x-\sqrt{x^2-6x+1}}{2x}.$$

For large $n$,

$$g_n=-\frac{1}{2}[x^{n+1}](x^2-6x+1)^{1/2}=2f_n\sim\frac{(n+1)^{-3/2}(3+2\sqrt{2})^{n+1/2}}{2^{3/4}\sqrt{\pi}}.$$

***<u>Example 2</u>—Legendre Polynomials:*** We require

$$g(x)=\frac{1}{\sqrt{x^2-2tx+1}}.$$

We take

$$\lambda=\frac{t^2-1}{4},\ \ b=t,\ \ \lambda_1=\lambda+\delta=\frac{t^2-1}{2},\ \text{and } b_0=b+\varepsilon=t.$$

***Triple Factorization:***

$$L=\left(\frac{1}{\sqrt{x^2-2tx+1}},\frac{2(1-tx-\sqrt{x^2-2tx+1})}{(t^2-1)x}\right)=P^tC_{(t^2-1)/4}F_{0,(t^2-1)/4},$$

$$S_L=\begin{bmatrix} t & 1 & & & & & \cdot \\ \frac{t^2-1}{2} & t & 1 & & & & \cdot \\ 0 & \frac{t^2-1}{4} & t & 1 & & & \cdot \\ 0 & 0 & \frac{t^2-1}{4} & t & 1 & & \cdot \\ 0 & 0 & 0 & \frac{t^2-1}{4} & t & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

***Recurrence Relations:*** Here $A=\frac{1-t^2}{4}$, $B=\frac{t(t^2-1)}{2}$, $C=\frac{1-t^2}{4}$. $(n+2)f_{n+1}=(2n+1)tf_n+(1-n)f_{n-1}$, for $n\geq 1$. $f_0=0, f_1=1$. $ng_n=(4n-3)tg_{n-1}+(3-2n)(1+2t^2)g_{n-2}+(4n-9)tg_{n-3}+(3-n)g_{n-4}$, for $n\geq 4$. $g_0=1, g_1=t, g_2=\frac{3}{2}t^2-\frac{1}{2}, g_3=\frac{5}{2}t^3-\frac{3t}{2}$.

***Asymptotics:*** We assume that $t^2>1$, so that the roots of $x^2-2tx+1=0$ are real. Denote these roots by $\hat{r}$ and $\tilde{r}$ with $|\hat{r}|<|\tilde{r}|$. We obtain

$$\begin{aligned}[x^n]f(x)&=-\frac{2}{t^2-1}[x^{n+1}](x^2-2tx+1)^{1/2}\\ &\sim\left(\frac{-2}{t^2-1}\right)\frac{1}{(\tilde{r})^{n+1}}\frac{(n+1)^{-3/2}}{-2\sqrt{\pi}}\left(1-\frac{\hat{r}}{\tilde{r}}\right)^{1/2}\\ &=\left(\frac{1}{t^2-1}\right)\frac{(\tilde{r})^n}{n+1}\sqrt{\frac{(\tilde{r})^2-1}{n\pi}};\end{aligned}$$

$$[x^n]g(x)=[x^n](x^2-2tx+1)^{-1/2}\sim\frac{(\tilde{r})^{n+1}}{\sqrt{n\pi((\tilde{r})^2-1)}}.$$

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