RECURSIONS AND PASCAL-TYPE TRIANGLES

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(Submitted October 1991)

INTRODUCTION

Triangular arrays of numbers similar to or derived from Pascal's triangle frequently appear in the mathematical literature. (See, for example, [3], [5], and [8].) The purpose of this paper is to study a generalization of the array in [8]. In section 1, recursion formulas for the row and diagonal row sums are derived. In section 2, the determinants of a set of matrices associated with the triangular array of [8] are calculated.

1. GENERAL PROPERTIES OF THE ARRAYS

Consider a family of triangular arrays of numbers, indexed by the reals. For each $a \in \mathbf{R}$, the array is a doubly infinite set of numbers $d(a; n, k)$; $n, k \in \mathbf{Z}$, such that:

a. $d(a; n, k) = 0, \ n < 0;$

b. $d(a; n, k) = 0, \ k < 0$ or $k > n;$

c. $d(a; 0, 0) = a,$

d. $d(a; 1, 0) = d(a; 1, 1) = 1;$ and

e. $d(a; n, k) = d(a; n-2, k-1) + d(a; n-1, k-1) + d(a; n-1, k), \ n \geq 2.$

The triangular array studied by Wong & Maddocks [8] corresponds to the case $a = 1$. Their general term $M_{k,r}$ corresponds to the term $d(1; k+r, r)$ here. Tables 1, 2, and 3 contain the initial rows for the arrays $d(1; n, k)$, $d(0; n, k)$, and the general array $d(a; n, k)$, respectively. As mentioned above, Table 1 appears in [8]. It also appears in [1].

TABLE 1. $d(1; n, k)$

1				
1	1			
1	3	3		
1	5	5	1	
1	7	13	7	1

TABLE 2. $d(0; n, k)$

0				
1	1			
1	2	1		
1	4	4	1	
1	6	10	6	1

TABLE 3. $d(a; n, k)$

a					
1	1				
1	$2+a$	1			
1	$4+a$	$4+a$	1		
1	$6+a$	$10+3a$	$6+a$	1	
1	$8+a$	$20+5a$	$20+5a$	$8+a$	1

An examination of these arrays reveals that, for $n \geq 2$,

$$d(a; n, k) = d(0; n, k) + a[d(1; n-2, k-2)].$$

Thus, calculations for any array $d(a;n,k)$ reduce to calculations on $d(0;n,k)$ and $d(1;n,k)$.

Definition 1: For fixed n, we call the sums

(1) $D(a;n)=\sum_{k=0}^{n} d(a;n,k)$; and

(2) $D^*(a;n)=\sum_{k=0}^{n}(-1)^k d(a;n,k)$

the *row sums* and the *alternating row sums*, respectively, of the array $d(a;n,k)$.

It is immediate that, for $n\geq 2$,

a. $D(a;n)=D(0;n)+a[D(1;n-2)]$; and

b. $D^*(a;n)=D^*(0;n)+(-a)[D^*(1;n-2)]$.

Theorem 1: The sequences $\{D(1;n)\}$ and $\{D(0;n)\}$ satisfy:

(a) $D(1;0)=1$; $D(1;1)=2$; and, for $n\geq 2$, $D(1;n)=2D(1;n-1)+D(1;n-2)$;

(b) $D^*(1;n)=\begin{cases}0, & n \text{ odd}, n>0,\\ (-1)^m, & n=2m, m\geq 0;\end{cases}$

(c) $D(0;0)=0$; $D(0;1)=2$; and, for $n\geq 1$, $D(0;n)=2D(0;n-1)+D(0;n-2)$; and

(d) For $n\geq 0$, $D^*(0;n)=0$.

Proof of (a): The proof is by induction. Obviously,

$$D(1;0)=1;\ D(1;1)=2;\ \text{and}\ D(1;2)=2D(1;1)+D(1;0).$$

Assume the proposition is true for $2\leq n<m$. For $n=m$,

$$\begin{aligned}D(1;m)&=\sum_{k=0}^{m} d(1;m,k)=\sum_{k=0}^{m}\{d(1;m-2,k-1)+d(1;m-1,k-1)+d(1;m-1,k)\}\\ &=\sum_{k=0}^{m} d(1;m-2,k-1)+\sum_{k=0}^{m}\{d(1;m-1,k-1)+d(1;m-1,k)\}.\end{aligned}$$

The first summation is $D(1;m-2)$. The second summation is

$$\begin{aligned}&\{d(1;m-1,-1)+d(1;m-1,0)\}+\{d(1;m-1,0)+d(1;m-1,1)\}\\ &\quad+\{d(1;m-1,1)+d(1;m-1,2)\}+\cdots+\{d(1;m-1,m-2)\\ &\quad+d(1;m-1,m-1)\}+\{d(1;m-1,m-1)+d(1;m-1,m)\}.\end{aligned}$$

Recall that $d(1;m-1,-1)=d(1;m-1,m)=0$. Regrouping, the summation becomes:

$$\begin{aligned}&2d(1;m-1,0)+2d(1;m-1,1)+\cdots+2d(1;m-1,m-2)\\ &\quad+2d(1;m-1,m-1)=2D(1;m-1).\end{aligned}$$

Thus, $D(1;m)=2D(1;m-1)+D(1;m-2)$.

The proofs of (b), (c), and (d) are similar. □

The recursions (a) and (c) identify the sequences $\{D(1;n)\}$ and $\{D(0;n)\}$ as Pell sequences [2]. The initial terms of the $D(1;n)$ sequences are: 1, 2, 5, 12, 29, 70, 169, ... This sequence is number 552 in Sloane [6]. The $D(0;n)$ sequence starts: 0, 2, 4, 10, 24, 58, ... The terms are all even. Dividing by 2 yields: 0, 1, 2, 5, 12, 29, 70, 169, ..., which is again Sloane's sequence 552.

Given Definition 1 and Theorem 1, a simple calculation yields

Corollary 1: The sequences $\{D(a;n)\}$ and $\{D^*(a;n)\}$ satisfy:

(a) $D(a;0)=a;\ D(a;1)=2;\ D(a;n)=2D(a;n-1)+D(a;n-2),\ n\geq 2.$

(b) $D^*(a;n)=\begin{cases}0, & n \text{ odd},\\ a(-1)^m, & n=2m.\end{cases}$

Definition 2: Sums of the form

(1) $\partial(a;n)=d(a;n,0)+d(a;n-1,1)+d(a;n-2,2)+\cdots$, and

(2) $\partial^*(a;n)=d(a;n,0)-d(a;n-1,1)+d(a;n-2,2)-d(a;n-3,3)+\cdots$, will be called *diagonal sums* and *alternating diagonal sums*, respectively, for the array $d(a;n,k)$.

Theorem 2: The diagonal sums $\partial(1;n)$ and $\partial(0;n)$ satisfy:

(a) $\partial(1;0)=\partial(1;1)=1;\ \partial(1;2)=2;$
and $\partial(1;n)=\partial(1;n-1)+\partial(1;n-2)+\partial(1;n-3);\ n\geq 3;$

(b) $\partial(0;0)=0;\ \partial(0;1)=1;\ \partial(0;2)=2;$
and $\partial(0;n)=\partial(0;n-1)+\partial(0;n-2)+\partial(0;n-3);\ n\geq 3.$

Proof: ***(a)*** Proved in [1] and [8]; ***(b)*** Direct calculation. □

The initial terms of the $\partial(1;n)$ sequence are: 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, ... This is Sloane's sequence 406 [6]. This sequence appeared in [1], [4], and [7], where it is called the *Tribonacci sequence*. The terms of $\partial(0;n)$ are: 0, 1, 2, 3, 6, 11, 20, 37, ...; Sloane's sequence 296. Both sequences have a three-term recursion; i.e., for both sequences, the recursion is of the form $s(n)=s(n-1)+s(n-2)+s(n-3), n\geq 3$. The difference between the two sequences reqults from different initial terms. Sequences with a three-term recurrence have been studied previously, e.g., [4], [7]. The recursion relations for both $\partial(0;n)$ and $\partial(1;n)$ can be written in matrix form [7].

Theorem 3: The alternating diagonal sums $\partial^*(1;n)$ and $\partial^*(0;n)$ satisfy the relations:

(a) $\partial^*(1;0)=\partial^*(1;1)=1;\ \partial^*(1;2)=0;$ and
$\partial^*(1;n)=\partial^*(1;n-1)-\partial^*(1;n-2)-\partial^*(1;n-3),\ n\geq 3.$

(b) $\partial^*(0;0)=0;\ \partial^*(0;1)=1;\ \partial^*(0;2)=0;$ and
$\partial^*(0;n)=\partial^*(0;n-1)-\partial^*(0;n-2)-\partial^*(0;n-3),\ n\geq 3.$

Corollary 2: The diagonal sums $\partial(a;n,k)$ satisfy

(a) $\partial(a;0)=a;\ \partial(a;1)=1;\ \partial(a;2)=2;$

(b) $\partial(a;n)=\partial(a;n-1)+\partial(a;n-2)+\partial(a;n-3);\ n\geq 3.$

The alternating diagonal sums $\partial^*(a;n)$ satisfy

(c) $\partial^*(a;0)=a;\ \partial^*(a;1)=1;\ \partial^*(a;2)=0;$

(d) $\partial^*(a;n)=\partial^*(a;n-1)-\partial^*(a;n-2)-\partial^*(a;n-3);\ n\geq 3.$

2. THE ASSOCIATED MATRICES

Rotate the array $d(1;n,k)$ counterclockwise so that the diagonals become rows and columns to produce the following infinite matrix:

$$\mathbf{M}=\begin{bmatrix}1 & 1 & 1 & 1 & 1 & \cdots\\ 1 & 3 & 5 & 7 & 9 & \cdots\\ 1 & 5 & 13 & 25 & 41 & \cdots\\ 1 & 7 & 25 & 63 & 129 & \cdots\\ 1 & 9 & 41 & 129 & 321 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots\end{bmatrix}$$

The recursion relations for the triangle translate to the following relations for the terms $m_{i,j}$ of the matrix:

a. $m_{1,j}=m_{i,1}=1$, for all i, j; and

b. $m_{i,j}=m_{i,j-1}+m_{i-1,j-1}+m_{i-1,j}, i>1, j>1.$

Let M_n be the $(n\times n)$-submatrix whose rows and columns are the first n rows and n columns of **M**, and $|M_n|$ the corresponding determinant.

Theorem 4: For $n\geq 1$, $|M_n|=2^{n(n-1)/2}$.

Proof: By induction. For $n=1$, the result is immediate.

For $k>1$, the matrix can be changed by elementary row and column operations so that, in block form,

$$M_k=\left[\begin{array}{c|c}1 & 0\\ \hline 0 & 2M_{k-1}\end{array}\right]$$

The rest follows. □

ACKNOWLEDGMENT

The author wishes to acknowledge the referees comments which substantially improved the paper and provided references the author was unfamiliar with.

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AMS numbers: 11B83; 11B39

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May 10, 1993

Dear Editor:

May I inform you that I have just read with interest the paper "On Extended Generalized Stirling Pairs" by A. G. Kyriakoussis, which appeared in *The Fibonacci Quarterly* **31.1** (1993):44-52. I wish to mention that Kyriakoussis' "EGSP" ("extended generalized Stirling pair") is actually a particular case included in the second class of extended "GSN" pairs considered in my paper "Theory and Application of Generalized Stirling Number Pairs," *J. Math. Res. and Exposition* **9** (1989):211-20. His first characterization theorem for "EGSP" is a special case of my Theorem 6 (*loc. cit.*). In fact, a basic result corresponding with his case appeared much earlier in the paper by J. L. Fields & M. E. H. Ismail, entitled "Polynomial Expansions," *Math. Comp.* **29** (1975):894-902.

Thank you for your attention.

Yours sincerely,

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