ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-483 *Proposed by James Nicholas Boots (deceased) & Lawrence Somer, The Catholic University of America, Washington, D.C.*

Let $m \geq 2$ be an integer such that

$$L_m \equiv 1 \pmod{m} \tag{1}$$

It is well known (see [1], p. 44) that if m is a prime, then (1) holds. It has been proved by H. J. A. Duparc [3] that there exist infinitely many composite integers, called Fibonacci pseudoprimes, such that (1) holds. It has also been proved in [2] and [4] that every Fibonacci pseudoprime is odd.

(i) Prove that

$$L_{m-1}^2 + L_{m-1} - 6 \equiv 0 \pmod{m}.$$

In particular, conclude that if m is prime, then $L_{m-1} \equiv 2$ or $-3 \pmod{m}$.

(ii) Prove that

$$F_{m-2} - L_{m-1}F_{m-1} \equiv 1 \pmod{m}.$$

References

1. R. D. Carmichael. "On the Numerical Factors of the Arithmetic Forms $\alpha^n \pm \beta^n$." *Ann. Math.*, Second Series **15** (1913):30-70.
2. A. Di Porto. "Nonexistence of Even Fibonacci Pseudoprimes of the 1st Kind." *The Fiboncci Quarterly* **31.2** (1993):173-77.
3. H. J. A. Duparc. "On Almost Primes of the Second Order," pp. 1-13. Amsterdam: Rapport ZW, 1955-013, Math. Center, 1955.
4. D. J. White, J. N. Hunt, & L. A. G. Dresel. "Uniform Huffman Sequences Do Not Exist." *Bull. London Math. Soc.* **9** (1977):193-98.

H-484 *Proposed by J. Rodriguez, Sonora, Mexico*

Find a strictly increasing infinite series of integer numbers such that, for any consecutive three of them, the Smarandache Function is neither increasing nor decreasing.

*Find the largest strictly increasing series of integer numbers for which the Smarandache Function is strictly decreasing.

H-485 ***Proposed by Paul S. Bruckman, Everett, WA***

If x is an unspecified large positive real number, obtain an asymptotic evaluation for the sum $S(x)$, where

$$S(x) = \sum_{p \le x} (-1)^{Z(p)}; \tag{1}$$

here, the p's are prime and $Z(p)$ is the Fibonacci entry-point of p (the smallest positive n such that $p | F_n$).

SOLUTIONS

Sum Problem

H-469 ***Proposed by H.-J. Seiffert, Berlin, Germany***
(Vol. 30, no. 3, August 1992)

Define the Fibonacci polynomials by

$$F_0(x) = 0,\ F_1(x) = 1,\ F_n(x) = xF_{n-1}(x) + F_{n-2}(x),\ \text{ for } n \ge 2.$$

Show that for all positive integers n and all positive reals x:

(a)
$$\frac{1}{F_{2n-1}(x)} = \frac{x^2+4}{2n-1} \sum_{k=0}^{2n-2} (-1)^{k+n+1} \frac{\cos\dfrac{k\pi}{2n-1}}{x^2 + 4\cos^2\dfrac{k\pi}{2n-1}};$$

(b)
$$\frac{1}{F_{2n}(x)} = \frac{x(x^2+4)}{4n} \sum_{k=0}^{2n-1} \frac{(-1)^{k+n}}{x^2 + 4\cos^2\dfrac{k\pi}{2n}}.$$

Solution by Paul S. Bruckman, Everett, WA

From the given recurrence relation and the initial conditions, we readily establish that $F_n(x)$ is a monic polynomial in x of degree $n-1$. In particular,

$$F_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{1}$$

where

$$\alpha = \alpha(x) = \tfrac{1}{2}(x + \sqrt{x^2+4}),\ \beta = \beta(x) = \tfrac{1}{2}(x - \sqrt{x^2+4}). \tag{2}$$

If we make the substitution

$$x = 2\sinh\theta, \tag{3}$$

we obtain $\alpha = e^\theta$, $\beta = -e^{-\theta}$, $2\cosh\theta = \sqrt{x^2+4}$. This leads to the alternative formulation:

$$F_{2n-1}(x) = \frac{\cosh(2n-1)\theta}{\cosh\theta}; \tag{4}$$

$$F_{2n}(x) = \frac{\sinh 2n\theta}{\cosh\theta}. \tag{5}$$

Proof of Part (a): We readily find the zeros of $F_{2n-1}(x)$ from (4) ($2n-2$ in number); we shall suppose that $n > 1$ initially.

Denoting these by x_k, we obtain:

$$x_k = y_k \text{ or } -y_k = \bar{y}_k, \text{ where } y_k = 2\sinh\frac{(2k-1)i\pi}{2(2n-1)} = 2i\sin\psi_k, \\ \text{and } \psi_k = \frac{(2k-1)\pi}{2(2n-1)}, \quad k = 1, 2, \ldots, n-1. \tag{6}$$

Note that the ψ_k's are distinct and $0 < \psi_k < \frac{1}{2}\pi$ for each k; thus, the x_k's are distinct. Therefore, the x_k's are simple poles of the function $1/F_{2n-1}(x)$. By the residue theory, we may find constants A_k such that

$$1/F_{2n-1}(x) = \sum_{k=1}^{n-1}\left(\frac{A_k}{x-y_k} + \frac{\overline{A}_k}{x+y_k}\right). \tag{7}$$

In fact, the A_k's are determined from the following:

$$A_k = \lim_{x\to y_k}\frac{x-y_k}{F_{2n-1}(x)}. \tag{8}$$

Then, applying L'Hopitâl's Rule,

$$\frac{1}{A_k} = \frac{d}{dx}F_{2n-1}(x)\bigg|_{x=y_k} = F'_{2n-1}(y_k).$$

Now $dx/d\theta = 2\cosh\theta$, which implies that $d\theta/dx = \frac{1}{2}\cosh\theta$. Hence, by (4), we obtain:

$$F'_{2n-1}(x) = \tfrac{1}{2}(\cosh\theta)^{-3}[(2n-1)\cosh\theta\sinh(2n-1)\theta - \sinh\theta\cosh(2n-1)\theta].$$

Then $\sinh(2n-1)i\psi_k = i\sin\left(k-\frac{1}{2}\right)\pi = -i(-1)^k$, and $\cosh(2n-1)i\psi_k = \cos\left(k-\frac{1}{2}\right)\pi = 0$. Thus,

$$F'_{2n-1}(y_k) = -i(-1)^k(2n-1)/2\cos^{-2}\psi_k, \text{ and } A_k = \frac{2i(-1)^k}{2n-1}\cos^2\psi_k.$$

Then, using (7), we obtain:

$$\begin{aligned} 1/F_{2n-1}(x) &= \frac{2i}{2n-1}\sum_{k=1}^{n-1}(-1)^k\cos^2\psi_k[(x-y_k)^{-1} - (x+y_k)^{-1}] \\ &= \frac{2i}{2n-1}\sum_{k=1}^{n-1}(-1)^k\cos^2\psi_k\cdot\frac{4i\sin\psi_k}{x^2+4\sin^2\psi_k} = \frac{8}{2n-1}\sum_{k=1}^{n-1}\frac{(-1)^{k+1}\cos^2\psi_k\sin\psi_k}{x^2+4\sin^2\psi_k} \end{aligned}$$

Substituting $n-k$ for k yields:

$$1/F_{2n-1}(x)=\frac{8}{2n-1}\sum_{k=1}^{n-1}\frac{(-1)^{n+k+1}\sin^2\varphi_k\cos\varphi_k}{x^2+4\cos^2\varphi_k},\ \text{where } \varphi_k=\frac{k\pi}{2n-1}.$$

Now, substituting $2n-1-k$ for k yields:

$$1/F_{2n-1}(x)=\frac{8}{2n-1}\sum_{k=n}^{2n-2}\frac{(-1)^{k+n+1}\sin^2\varphi_k\cos\varphi_k}{x^2+4\cos^2\varphi_k}.$$

By addition, we obtain:

$$1/F_{2n-1}(x)=\frac{4}{2n-1}\sum_{k=1}^{2n-2}\frac{(-1)^{k+n+1}\sin^2\varphi_k\cos\varphi_k}{x^2+4\cos^2\varphi_k}. \tag{9}$$

We may also include the term for $k=0$ in the sum indicated in (9), since this term vanishes. Note that we have the following series manipulation:

$$1/F_{2n-1}(x)=(2n-1)^{-1}\sum_{k=0}^{2n-2}(-1)^{k+n+1}\cos\varphi_k\cdot\frac{x^2+4-x^2-4\cos^2\varphi_k}{x^2+4\cos^2\varphi_k},$$

or

$$1/F_{2n-1}(x)=\frac{x^2+4}{2n-1}\sum_{k=0}^{2n-2}(-1)^{k+n+1}\frac{\cos\varphi_k}{x^2+4\cos^2\varphi_k}+\frac{(-1)^n}{2n-1}S_n, \tag{10}$$

where

$$S_n=\sum_{k=0}^{2n-2}(-1)^k\cos\varphi_k. \tag{11}$$

Comparing (10) with the desired answer to part (a), we see that it only remains to show that $S_n=0$. This is readily determined as follows:

$$S_n=\operatorname{Re}\sum_{k=0}^{2n-2}(-1)^k\exp(ik\pi/(2n-1))$$

$$=\operatorname{Re}\left\{\frac{1-(-\exp(i\pi/(2n-1))^{2n-1}}{1+\exp(i\pi/(2n-1))}\right\}=\operatorname{Re}\left\{\frac{1+\exp(i\pi)}{1+\exp(i\pi/(2n-1))}\right\}=0.$$

Thus, part (a) is proved for $n>1$. Also, we see that the indicated formula gives the correct expression for $n=1$. This completes the proof of part (a).

Proof of Part (b): We suppose $n>0$. From (5), we find that $F_{2n}(x)$ has $2n-1$ simple zeros, given by $z_0=0$, z_k or $-z_k=\bar{z}_k$, where $z_k=2\sinh(ki\pi/2n)=2i\sinh\xi_k$, and $\xi_k=k\pi/2n$, $k=1,2,\dots,n-1$. As before, we find that

$$1/F_{2n}(x)=B_0/x+\sum_{k=1}^{n-1}\left(\frac{B_k}{x-z_k}+\frac{\overline{B}_k}{x+z_k}\right), \tag{12}$$

where

$$B_k=\lim_{x\to z_k}\frac{x-z_k}{F_{2n}(x)}=1/F'_{2n}(z_k),\ k=0,1,\dots,n-1. \tag{13}$$

We find that $F'_{2n}(x) = \frac{1}{2}(\cosh\theta)^{-3}[2n\cosh\theta\cosh 2n\theta - \sinh\theta\sinh 2n\theta]$, using (5). Then $\cosh 2in\xi_k = \cos k\pi = (-1)^k$ and $\sinh 2in\xi_k = i\sin k\pi = 0$; hence, $F'_{2n}(z_k) = n(-1)^k\cos^{-2}\xi_k$, and $B_k = \frac{1}{n}(-1)^k\cos^2\xi_k$ (note that $B_0 = 1/n$). Then

$$1/F_{2n}(x) = \frac{1}{nx} + \frac{1}{n}\sum_{k=1}^{n-1}(-1)^k\cos^2\xi_k[(x-2i\sin\xi_k)^{-1} + (x+2i\sin\xi_k)^{-1}]$$

$$= \frac{1}{nx} + \frac{2x}{n}\sum_{k=1}^{n-1}\frac{(-1)^k\cos^2\xi_k}{x^2+4\sin^2\xi_k}$$

Replacing k by $n-k$ yields:

$$1/F_{2n}(x) = \frac{1}{nx} + \frac{2x}{n}\sum_{k=1}^{n-1}(-1)^{n+k}\frac{\sin^2\xi_k}{x^2+4\cos^2\xi_k}.$$

Now, replacing k by $2n-k$ also yields:

$$1/F_{2n}(x) = \frac{1}{nx} + \frac{2x}{n}\sum_{k=n+1}^{2n-1}(-1)^{n+k}\frac{\sin^2\xi_k}{x^2+4\cos^2\xi_k}.$$

Then, adding the last two expressions, we obtain:

$$2/F_{2n}(x) = \frac{2}{nx} + \frac{2x}{n}\sum_{k=0}^{2n-1}U_k - \frac{2x}{n}\cdot x^{-2},$$

where

$$U_k = (-1)^{k+n}\frac{\sin^2\xi_k}{x^2+4\cos^2\xi_k}.$$

Thus, we find that

$$1/F_{2n}(x) = \frac{x}{4n}\sum_{k=0}^{2n-1}(-1)^{k+n}\frac{(x^2+4-x^2-4\cos^2\xi_k)}{x^2+4\cos^2\xi_k} = \frac{x(x^2+4)}{4n}\sum_{k=0}^{2n-1}V_k - (-1)^n\frac{x}{4n}T_n,$$

where

$$V_k = (-1)^{k+n}(x^2+4\cos^2 k\pi/2n)^{-1}, \text{ and } T_n = \sum_{k=0}^{2n-1}(-1)^k.$$

Clearly, $T_n = 0$. Therefore, the last result reduces to the expression given in part (b). Q.E.D.

Also solved by Hans Kappus and the proposer.

Characteristically Common

H-470 ***Proposed by Paul S. Bruckman, Everett, WA***
(Vol. 30, no. 3, August 1992)

Please see the issue of *The Fibonacci Quarterly* shown above for a complete presentation of this lengthy problem proposal.

Solution by the proposer

Proof of Part (A): We begin with the definition of $p_r(z)$, namely, $p_r(z) = |zI_r - U_1^{(r)}|$, where I_r is the $r \times r$ identity matrix. Thus,

$$p_r(z) = \begin{vmatrix} z-a_0 & -a_1 & -a_2 & \cdots & -a_{r-1} \\ -1 & z & 0 & \cdots & 0 \\ 0 & -1 & z & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & z \end{vmatrix}$$

Expanding along the last column, we obtain: $p_r(z) = (-1)^r a_{r-1} A_{r-1}(z) + zp_{r-1}(z)$, where $A_{r-1}(z)$ is the determinant of the $(r-1) \times (r-1)$ matrix whose elements a_{ij} are defined by: $a_{ij} = z\delta_{i+1,\,j} - \delta_{i,\,j}$; thus, this matrix is upper triangular, and so $A_{r-1}(x) = (-1)^{r-1}$, the product of the diagonal entries. Hence, $p_r(z) = zp_{r-1}(z) - a_{r-1}$. We note that $p_1(z) = z - a_0$; thus, $p_2(z) = z(z-a_0) - a_1 = z^2 - a_0 z - a_1$; $p_3(z) = z(z^2 - a_0 z - a_1) - a_2 = z^3 - a_0 z^2 - a_1 z - a_2$; and we see, in general, that we have:

$$p_r(z) = G_r(z). \quad \square \qquad (*)$$

Proof of Part (C): We suppose $r > 1$. Clearly, the desired relation is valid for $n = 1$. Suppose it is valid for some value of $n \ge 1$. Then $(U_1^{(r)})^n H_1^{(r)} = U_1^{(r)} (U_1^{(r)})^{n-1} H_1^{(r)}$ or, by the inductive hypothesis:

$$(U_1^{(r)})^n H_1^{(r)} = U_1^{(r)} H_n^{(r)} \text{ (for this special value of } n). \qquad (6)$$

Premultiplication of the j^{th} column of $H_n^{(r)}$ by the i^{th} row of $U_1^{(r)}$ replaces $H_{n+r-i,\,j}^{(r)}$ by $H_{n+r-i+1,\,j}^{(r)}$, which is clear, using (5), if $i > 1$; however, this is also true for $i = 1$, since $H_{n+r-1,\,j}^{(r)}$ is then transformed to $\sum_{k=0}^{r-1} a_k H_{n+r-1-k,\,j}^{(r)}$, which is equal to $H_{n+r,\,j}^{(r)}$, by the recurrence $G_r(E)(H_{n,\,j}^{(r)}) = 0$. Thus, we see that $U_1^{(r)} H_n^{(r)} = H_{n+1}^{(r)}$; it follows from (6) that $(U_1^{(r)})^n H_1^{(r)} = H_{n+1}^{(r)}$, which is the statement of part (C) for $n+1$. The result then follows by induction. $\square$

The proof of part (B) will appear in the May 1994 issue of this Quarterly.

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