ON A CONJECTURE OF PIERO FILIPPONI

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1. INTRODUCTION

Let us define a generalized Lucas sequence $\{H_n(m)\}$ by

$$H_n(m) = H_{n-1}(m) + mH_{n-2}(m),\ H_0(m) = 2,\ H_1(m) = 1, \tag{1}$$

where $m \geq 1$ is a natural number.

In a communication that appeared in a recent issue of this journal [1], P. Filipponi showed that

$$H_{p^s}(p) \equiv 1 \pmod{p^s} \tag{2}$$

where p is an odd prime, and he proposed also the following Conjecture:

$$H_{p^s}(p-1) \equiv 1 \pmod{p^s} \tag{3}$$

where $p \geq 5$ is a prime number.

Following a method introduced by Lucas ([2], p. 209; [3]), we shall prove here generalizations of (2) and (3), namely,

Theorem 1: If $p \geq 1$ is a *natural* number, and if $m \equiv 0 \pmod p$, then

$$H_{p^s}(m) \equiv 1 \pmod{p^{s+1}},\ s \geq 0.$$

Theorem 2: If $p \geq 5$ is a *prime* number and if $m \equiv -1 \pmod p$, then

$$H_{p^s}(m) \equiv 1 \pmod{p^{s+1}},\ s \geq 0.$$

2. PRELIMINARIES

Let us recall Waring's formula

$$x^p + y^p = (x+y)^p + p\sum_{k=1}^{[p/2]} (-1)^k C_{p,k} (xy)^k (x+y)^{p-2k},$$

where p is a natural integer, and

$$C_{p,k} = \frac{1}{p-k}\binom{p-k}{k} = \frac{1}{k}\binom{p-k-1}{k-1},\ \text{for } 1 \leq k \leq [p/2].$$

In our proofs, we shall need the following three lemmas.

Lemma 1: ***(i)*** If p is a natural integer, then $p, C_{p,k}$ is integral;

(ii) If p is a prime, then $C_{p,k}$ is integral.

Proof: ***(i)*** The result follows from the relation

$$pC_{p,k} = \binom{p-k}{k} + \binom{p-k-1}{k-1}.$$

(ii) From the relation

$$k\binom{p-k}{k} = (p-k)\binom{p-k-1}{k-1},$$

and since $\gcd(k, p-k) = 1$, it is clear that k divides $\binom{p-k-1}{k-1}$.

Lemma 2: If $p \equiv \pm 1 \pmod 6$ is a natural number, then $\Sigma_{k=1}^{[p/2]}(-1)^k C_{p,k} = 0$..

Proof: Let us put $x = e^{i\pi/3}$ and $y = e^{-i\pi/3}$ in Waring's formula to get

$$2\cos p\pi/3 = 1 + p\sum_{k=1}^{[p/2]}(-1)^k C_{p,k},$$

and the conclusion follows from this, since $2\cos p\pi/3 = 1$, when $p \equiv \pm 1 \pmod 6$.

Lemma 3: If p is an odd integer, then $(\ell p - 1)^{p^s} \equiv -1 \pmod{p^{s+1}}$, $\ell \geq 0$.

Proof: The statement clearly holds for $s = 0$. Supposing that $(\ell p-1)^{p^s} = -1 + Ap^{s+1}$, where A is an integer, one can write

$$\begin{aligned}(\ell p-1)^{p^{s+1}} &= (-1 + Ap^{s+1})^p \\ &= (-1)^p + \binom{p}{1}(-1)^{p-1}Ap^{s+1} + \binom{p}{2}(-1)^{p-2}A^2p^{2s+2} + \cdots + A^p p^{p(s+1)} \equiv -1 \pmod{p^{s+2}},\end{aligned}$$

since p is odd and $\binom{p}{1} = p$.

Let us return to the recurrence relation (1). We have $H_n(m) = \alpha_m^n + \beta_m^n$, where α_m and β_m are the real numbers such that $\alpha_m + \beta_m = 1$ and $\alpha_m\beta_m = -m$. Following Lucas ([2], p. 212), we replace x (resp. y) by $\alpha_m^{p^s}$ (resp. $\beta_m^{p^s}$) in Waring's formula to get

$$H_{p^{s+1}}(m) = H_{p^s}^p(m) + p\sum_{k=1}^{[p/2]}(-1)^{k(1+p^s)}C_{p,k}m^{kp^s}H_{p^s}^{p-2k}(m), \tag{4}$$

where p is a natural number.

3. PROOF OF THEOREM 1

The case $p = 1$ needs no comment, since $H_1 = 1$, so we suppose in the sequel that $p \geq 2$, and thus that $[p/2] \geq 1$.

Let us write H_n instead of $H_n(m)$ in (4), to get

$$H_{p^{s+1}} = H_{p^s}^p + (-1)^{1+p^s}pm^{p^s}H_{p^s}^{p-2} + \sum_{k=2}^{[p/2]}(-1)^{k(1+p^s)}pC_{p,k}m^{kp^s}H_{p^s}^{p-2k}, \tag{5}$$

since $C_{p,1}=1$. Notice that the last sum is empty for $p=2$ and $p=3$ and that $pC_{p,k}$ is an integer, by Lemma 1(i).

We proceed by induction upon s. The statement clearly holds for $s=0$ since $H_1=1$.

Now, let us suppose that

$$H_{p^s}\equiv 1 \pmod{p^{s+1}}.$$

By using an argument similar to the one used in Lemma 3, one can easily deduce from this that

$$H_{p^s}^p\equiv 1 \pmod{p^{s+2}}. \tag{6}$$

Next we have, for every $s\geq 0$ and every $p\geq 2$, $p^s\geq 2^s\geq s+1$, and thus

(a) $pm^{p^s}\equiv 0 \pmod{p^{s+2}}$.

On the other hand we have, for every $k\geq 2, kp^s\geq 22^s=2^{s+1}\geq s+2$, and thus

(b) $m^{kp^s}\equiv 0 \pmod{p^{s+2}}$.

Now, by using (6), (a), and (b) in (5), we have

$$H_{p^{s+1}}\equiv 1 \pmod{p^{s+2}}.$$

This concludes the proof of Theorem 1.

4. PROOF OF THEOREM 2

We suppose now that $p\geq 5$ is a prime number, and thus that $p\equiv\pm 1 \pmod 6$. Let us put $m=\ell p-1$ in (4) and write H_n instead of $H_n(\ell p-1)$ to obtain

$$H_{p^{s+1}}=H_{p^s}^p+p\sum_{k=1}^{[p/2]}C_{p,k}(\ell p-1)^{kp^s}H_{p^s}^{p-2k}. \tag{7}$$

We proceed by induction on s. The statement clearly holds for $s=0$, since $H_1=1$. Supposing that $H_{p^s}\equiv 1 \pmod{p^{s+1}}$, we obtain

$$H_{p^s}^{p-2k}\equiv 1 \pmod{p^{s+1}}, \text{ for } 1\leq k\leq [p/2], \tag{8}$$

and

$$H_{p^s}^p\equiv 1 \pmod{p^{s+2}}. \tag{9}$$

On the other hand, we have, by Lemma 3,

$$(\ell p-1)^{kp^s}\equiv (-1)^k \pmod{p^{s+1}}. \tag{10}$$

By Lemma 1(ii), $C_{p,k}$ is an integer, and by (8), (10), and Lemma 2, we obtain

$$\sum_{k=1}^{[p/2]}C_{p,k}(\ell p-1)^{kp^s}H_{p^s}^{p-2k}\equiv\sum_{k=1}^{[p/2]}C_{p,k}(-1)^k\equiv 0 \pmod{p^{s+1}}. \tag{11}$$

Now, by (7), (9), and (11), it is clear that $H_{p^{s+1}}\equiv 1 \pmod{p^{s+2}}$. This concludes the proof of Theorem 2.

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REFERENCES

1. P. Filipponi. "A Note on a Class of Lucas Sequences." *The Fibonacci Quarterly* **29.3** (1991):256-63.
2. E. Lucas. "Théorie des fonctions numériques simplement périodiques." *Amer. J. Math.* **1** (1878):184-220, 289-321.
3. E. Lucas. *The Theory of Simply Periodic Numerical Functions.* Tr. from French by Sydney Karavitz, ed. Douglas Lind. Santa Clara, Calif.: The Fibonacci Association, 1969.

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