ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to 72717.3515@compuserve.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

The Pell numbers P_n and their associated numbers Q_n satisfy

$$P_{n+2} = 2P_{n+1} + P_n,\ P_0 = 0,\ P_1 = 1;$$
$$Q_{n+2} = 2Q_{n+1} + Q_n,\ Q_0 = 1,\ Q_1 = 1.$$

If $p = 1+\sqrt{2}$ and $q = 1-\sqrt{2}$, then $P_n = (p^n - q^n)/\sqrt{8}$ and $Q_n = (p^n + q^n)/2$. The Pell-Lucas numbers, R_n, are given by $R_n = 2Q_n$. For more information about Pell numbers, see Marjorie Bicknell, "A Primer on the Pell Sequences and Related Sequences," *The Fibonacci Quarterly* **13.4** (1975):345-49.

PROBLEMS PROPOSED IN THIS ISSUE

The problems in this issue all involve Pell numbers. See the basic formulas above for definitions.

B-754 ***Proposed by Joseph J. Kostal, University of Illinois at Chicago, IL***

Find closed form expressions for

$$\sum_{k=1}^{n} P_k \quad \text{and} \quad \sum_{k=1}^{n} Q_k .$$

B-755 ***Proposed by Russell Jay Hendel, Morris College, Sumter, SC***

Find all nonnegative integers m and n such that $P_n = Q_m$.

B-756 ***Proposed by the editor***

Find a formula expressing P_n in terms of Fibonacci and/or Lucas numbers.

B-757 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Show that for $n > 0$,

(a) $$P_{3n-1} \equiv F_{n+2} \pmod{13},$$

(b) $$P_{3n+1} \equiv (-1)^{\lfloor (n+1)/2 \rfloor} F_{4n-1} \pmod{7}.$$

B-758 ***Proposed by Russell Euler, Northwest Missouri State University, Maryville, MO***

Evaluate

$$\sum_{k=0}^{\infty} \frac{k 2^k Q_k}{5^k}.$$

B-759 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Show that for all positive integers k and all nonnegative integers n,

$$\sum_{j=0}^{n} F_{k(j+1)} P_{k(n-j+1)} = \frac{F_k P_{k(n+2)} - P_k F_{k(n+2)}}{2Q_k - L_k}.$$

SOLUTIONS

A 7-Term Arithmetic Progression

B-724 ***Proposed by Larry Taylor, Rego Park, NY***
(Vol. 30, no. 4, November 1992)

Let n be a positive integer. Prove that the numbers $L_{n-1}L_{n+1}, 5F_n^2, L_{3n}/L_n, L_{2n}, F_{3n}/F_n, L_n^2, 5F_{n-1}F_{n+1}$ are in arithmetic progression and find the common difference.

Solution by Y. H. Harris Kwong, SUNY at Fredonia, NY

Using the basic formulas $F_n = (\alpha^n - \beta^n)/\sqrt{5}, L_n = \alpha^n + \beta^n$ and the identities $\alpha^2 + \beta^2 = 3$, $\alpha\beta = -1$, it is easy to show that the seven numbers form the arithmetic progression $L_{2n} + k(-1)^n$, $k = -3, -2, \ldots, 2, 3$, with common difference $(-1)^n$.

For example,

$$L_{n-1}L_{n+1} = (\alpha^{n-1} + \beta^{n-1})(\alpha^{n+1} + \beta^{n+1}) = \alpha^{2n} + \beta^{2n} + (\alpha\beta)^{n-1}(\alpha^2 + \beta^2) = L_{2n} - 3(-1)^n.$$

The other parts follow in a similar manner.

Most solutions were similar. The arithmetic progression can also be expressed as $L_n^2 + k(-1)^n$, $k = -5, -4, -3, \ldots, 1$.

Also solved by M. A. Ballieu, Seung-Jin Bang, Brian D. Beasley, Scott H. Brown, Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, Russell Euler, Piero Filipponi, C. Georghiou, Pentti Haukkanen, John Ivie, Russell Jay Hendel, Joseph J. Kostal, Carl Libis, Graham Lord, Igor Ol. Popov, Bob Prielipp, H.-J. Seiffert, A. G. Shannon, Sahib Singh, Lawrence Somer, Ralph Thomas, and the proposer.

An Infinite Set of Right Triangles

B-725 ***Proposed by Russell Jay Hendel, Patchogue, NY* and *Herta T. Freitag, Roanoke, VA (Vol. 30, no. 4, November 1992)***

(a) Find an infinite set of right triangles each of which has a hypotenuse whose length is a Fibonacci number and an area that is the product of four Fibonacci numbers.

(b) Find an infinite set of right triangles each of which has a hypotenuse whose length is the product of two Fibonacci numbers and an area that is the product of four Lucas numbers.

Solution by the proposers

Recall that $A = x^2 - y^2$, $B = 2xy$, and $C = x^2 + y^2$ form a Pythagorean triangle with area $xy(x-y)(x+y)$.

(a) Let $x = F_n$, $y = F_{n-1}$, and use the fact that $F_n^2 + F_{n-1}^2 = F_{2n-1}$ (see [1]).

(b) Let $x = L_{n+1}$, $y = L_n$, and use the fact that $L_n^2 + L_{n+1}^2 = F_5 F_{2n+1}$ (see [1]).

Reference:

1. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section: Theory and Applications.* Chichester: Ellis Horwood Ltd., 1989, p. 29.

The proposers also found an infinite set of right triangles whose hypotenuse is a Pell number and whose area is the product of four Pell numbers. Shannon noted that if (H_n) is any sequence that satisfies the recurrence $H_n = H_{n-1} + H_{n-2}$, then the triangle with sides $H_n H_{n+3}$, $2H_{n+1}H_{n+2}$, and $2H_{n+1}H_{n+2} + H_n^2$ is a Pythagorean triangle with area $H_n H_{n+1} H_{n+2} H_{n+3}$. However, he was unable to put the length of the hypotenuse, $2H_{n+1}H_{n+2} + H_n^2$, into a simpler form.

Also solved by Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, Daniel C. Fielder & Cecil O. Alford, C. Georghiou, Igor Ol. Popov, Bob Prielipp, H.-J. Seiffert, A. G. Shannon, Sahib Singh, and Lawrence Somer.

A Diverting Sum

B-726 ***Proposed by Florentin Smarandache, Phoenix, AZ (Vol. 30, no. 4, November 1992)***

Let $d_n = P_{n+1} - P_n$, $n = 1, 2, 3, \ldots$, where P_n is the n^{th} prime. Does the series

$$\sum_{n=1}^{\infty} \frac{1}{d_n}$$

converge?

Solution by C. Georghiou, University of Patras, Greece

The series diverges! This can be seen by noticing that

$$d_n = P_{n+1} - P_n < P_{n+1}.$$

We use the well-known fact ([1], p. 17) that the series of the reciprocals of the prime numbers diverges and the standard Comparison Test ([2], p. 777) which says that if $\sum a_k$ diverges and $b_k > a_k > 0$ for all k, then $\sum b_k$ diverges.

References:
1. G. H. Hardy & E. M. Wright. *An Introduction to the Theory of Numbers*. 5th ed. Oxford: Oxford University Press, 1979.
2. George B. Thomas. *Calculus and Analytic Geometry*. 3rd ed. Reading, MA.: Addison-Wesley, 1960.

Several solvers invoked Bertrand's Postulate ([1], p. 343). Seiffert asks if the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{d_n}$$

diverges. He notes that this would follow from the twin prime conjecture ([1], p. 5).

Also solved by Charles E. Chace & Russell Jay Hendel, Leonard A. G. Dresel, Piero Filipponi, H.-J. Seiffert, Sahib Singh, and the proposer.

It's a Tanh

B-727 ***Proposed by Ioan Sadoveanu, Ellensburg, WA (Vol. 30, no. 4, November 1992)***

Find the general term of the sequence (a_n) defined by the recurrence

$$a_{n+2} = \frac{a_{n+1} + a_n}{1 + a_{n+1}a_n}$$

with initial values $a_0 = 0$ and $a_1 = (e^2 - 1)/(e^2 + 1)$, where e is the base of natural logarithms.

Solution by C. Georghiou, University of Patras, Greece

Let b_n be defined by $a_n = \tanh b_n$. This is possible because the hyperbolic tangent defined on $[0, \infty)$ and valued in $[0, 1)$ is a one-to-one function. Note that $a_1 = (e^1 - e^{-1})/(e^1 + e^{-1}) = \tanh 1$ from the formula $\tanh x = (e^x - e^{-x})/(e^x + e^{-x})$. From the well-known formula

$$\tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y},$$

we get $b_{n+2} = b_{n+1} + b_n$ with $b_0 = 0$ and $b_1 = 1$. Therefore, $b_n = F_n$, and the answer to the problem is $a_n = \tanh F_n$. (The tanh formulas can be found on page 24 in [1].)

Reference:
1. I. S. Gradshteyn & I. M. Ryzhik. *Tables of Integrals, Series and Products*. San Diego, CA: Academic Press, 1980.

Several solvers gave the equivalent answer $a_n = (e^{2F_n} - 1)/(e^{2F_n} + 1)$. In this form, Lord notes that "e" could be any constant. The proposer solved the problem for a_0 and a_1 being arbitrary constants in (–1, 1), but the answer is a complicated expression.

Also solved by Tareq Alnaffouri, Richard André-Jeannin, Seung-Jin Bang, Brian D. Beasley, Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, Russell Jay Hendel, Y. H. Harris Kwong, Carl Libis, Graham Lord, Samih A. Obaid, Igor Ol. Popov, H.-J. Seiffert, A. G. Shannon, Sahib Singh, Ralph Thomas, and the proposer.

When Does Mod p Imply Mod p^2?

B-728 ***Proposed by Leonard A. G. Dresel, Reading, England (Vol. 30, no. 4, November 1992)***

If $p > 5$ is a prime and n is an even integer, prove that

(a) if $L_n \equiv 2 \pmod{p}$, then $L_n \equiv 2 \pmod{p^2}$;

(b) if $L_n \equiv -2 \pmod{p}$, then $L_n \equiv -2 \pmod{p^2}$.

Solution by A. G. Shannon, University of Technology, Sydney, Australia

We have $L_n = \alpha^n + \beta^n$ (with $\alpha\beta = -1$) and $n = 2k$. Let $x = L_n \pm 2$. We want to show that if $p|x$, then $p^2|x$. Note that $x = L_n \pm 2 = (\alpha^k \pm (-1)^k \beta^k)^2$. If $p|x$, then $p|L_k^2$ or $p|5F_k^2$. In either case, since p is a prime larger than 5, we must have $p^2|x$.

Several solvers noted that p could be any prime not equal to 5.

Also solved by Richard André-Jeannin, Paul S. Bruckman, Russell Jay Hendel, Y. H. Harris Kwong, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, and the proposer.

Binet to the Rescue Again

B-729 ***Proposed by Lawrence Somer, Catholic University of America, Washington, DC (Vol. 30, no. 4, November 1992)***

Let (H_n) denote the second-order linear recurrence defined by $H_{n+2} = aH_{n+1} + bH_n$, where $H_0 = 0, H_1 = 1$, and a and b are integers. Let p be a prime such that $p \nmid b$. Let k be the least positive integer such that $H_k \equiv 0 \pmod{p}$. (It is well known that k exists.) If $H_n \not\equiv 0 \pmod{p}$, let $R_n \equiv H_{n+1}H_n^{-1} \pmod{p}$.

(a) Show that $R_n + R_{k-n} \equiv a \pmod{p}$ for $1 \le n \le k-1$.

(b) Show that $R_n R_{k-n-1} \equiv -b \pmod{p}$ for $1 \le n \le k-2$.

Solution by A. G. Shannon, University of Technology, Sydney, Australia, and by Y. H. Harris Kwong, SUNY at Fredonia, NY (independently).

We will first prove the identity

$$H_{n+1}H_{k-n} + H_n H_{k-n+1} = H_k + aH_n H_{k-n}, \qquad (*)$$

which is valid for all integers n and k. The Binet form [1] giving the explicit value for H_n is

$$H_n = \frac{A^n - B^n}{A - B},$$

where $A = (a + \sqrt{a^2 + 4b})/2$ and $B = (a - \sqrt{a^2 + 4b})/2$ are the roots of $x^2 = ax + b$. Straightforward algebra allows us to check the identity

$$\begin{aligned}&(A^{n+1} - B^{n+1})(A^{k-n} - B^{k-n}) + (A^n - B^n)(A^{k-n+1} - B^{k-n+1})\\ &= (A - B)(A^k - B^k) + (A + B)(A^n - B^n)(A^{k-n} - B^{k-n})\end{aligned}$$

from which $(*)$ follows since $A + B = a$.

(a) From the definition of R_n, we have $H_n R_n \equiv H_{n+1} \pmod{p}$. From $(*)$ and the fact that $H_k \equiv 0 \pmod{p}$, we get

$$\begin{aligned} aH_nH_{k-n} &\equiv H_{n+1}H_{k-n} + H_nH_{k-n+1} && \pmod p \\ &\equiv H_nH_{k-n}R_n + H_nH_{k-n}R_{k-n} && \pmod p \end{aligned}$$

and the result follows since $H_n \not\equiv 0 \pmod p$ for $1 \le n \le k-1$.

(b) Using part (a) and the definition of R_n gives

$$\begin{aligned} H_nH_{k-n-1}R_nR_{k-n-1} &\equiv H_{n+1}H_{k-n} && \pmod p \\ &\equiv H_n(aH_{k-n} - H_{k-n+1}) && \pmod p \\ &\equiv -bH_nH_{k-n-1} && \pmod p \end{aligned}$$

and again the result follows for primes p that do not divide b.

Reference:

1. Ivan Niven, Herbert S. Zuckerman, & Hugh L. Montgomery. *An Introduction to the Theory of Numbers.* 5th ed. New York: Wiley & Sons, 1991, p. 199, Th. 4.10.

Also solved by Paul S. Bruckman, Leonard A. G. Dresel, H.-J. Seiffert, and the proposer.

A Golden Quadratic

B-730 ***Proposed by Herta T. Freitag, Roanoke, VA***
(Vol. 31, no. 1, February 1993)

For $n \ge 0$, express the larger root of $x^2 - L_nx + (-1)^n = 0$ in terms of α, the larger root of $x^2 - x - |(-1)^n| = 0$.

Solution by F. J. Flanigan, San Jose State University, San Jose, CA; Sahib Singh, Clarion University of Pennsylvania, Clarion, PA; **and** ***A. N. 't Woord, Eindhoven University of Technology, the Netherlands (independently)***

From $L_n = \alpha^n + \beta^n$ and $\alpha\beta = -1$, we have

$$x^2 - L_nx + (-1)^n = x^2 - (\alpha^n + \beta^n)x + (\alpha\beta)^n = (x - \alpha^n)(x - \beta^n),$$

and since $\alpha > |\beta| > 0$, the largest root is α^n.

Haukkanen notes that the roots of $x^2 + L_nx + (-1)^n = 0$ *are* $x = -\beta^n$ *and* $x = -\alpha^n$*; the roots of* $x^2 - \sqrt{5}F_nx - (-1)^n = 0$ *are* $x = \alpha^n$ *and* $x = -\beta^n$*; and the roots of* $x^2 + \sqrt{5}F_nx - (-1)^n = 0$ *are* $x = \beta^n$ *and* $x = -\alpha^n$.

Also solved by Richard André-Jeannin, M. A. Ballieu, Seung-Jin Bang, Brian D. Beasley, Paul S. Bruckman, Joseph E. Chance, the Con Amore Problem Group, Elizabeth Desautel & Charles K. Cook, Leonard A. G. Dresel, Russell Euler, Pentti Haukkanen, Russell Jay Hendel, John Ivie, Ed Kornt-ved, Carl Libis, Don Redmond, H.-J. Seiffert, Lawrence Somer, J. Suck, Ralph Thomas, and the proposer.

Errata: Russell Jay Hendel was inadvertently omitted as a solver for Problems B-718, B-719, B-720 and B-722.

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