ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-486 *Proposed by Piero Filipponi, Rome, Italy*

Let the terms of the sequence $\{Q_k\}$ be defined by the second-order recurrence relation $Q_k = 2Q_{k-1} + Q_{k-2}$ with initial conditions $Q_0 = Q_1 = 1$. Find restrictions on the positive integers n and m for

$$T(n, m) = \sum_{k=1}^{\infty} \frac{k^2 n^k Q_k}{m^k}$$

to converge, and, under these restrictions, evaluate this sum. Moreover, find the set of all couples (n_i, m_i) for which $T(n_i, m_i)$ is an integer.

H-487 *Proposed by Stanley Rabinowitz, MathPro Press, Westford, MA*

Suppose H_n satisfies a second-order linear recurrence with constant coefficients. Let $\{a_i\}$ and $\{b_i\}$, $i = 1, 2, \ldots, r$, be integer constants and let $f(x_0, x_1, x_2, \ldots, x_r)$ be a polynomial with integer coefficients. If the expression

$$f((-1)^n, H_{a_1 n+b_1}, H_{a_2 n+b_2}, \ldots, H_{a_r n+b_r})$$

vanishes for all integers $n > N$, prove that the expression vanishes for all integral n.

[As a special case, if an identity involving Fibonacci and Lucas numbers is true for all positive subscripts, then it must also be true for all negative subscripts as well.]

SOLUTIONS

Characteristically Common

H-470 *Proposed by Paul S. Bruckman, Everett, WA*
(Vol. 30, no. 3, August 1992)

Please see the issue of *The Fibonacci Quarterly* shown above for a complete presentation of this lengthy problem proposal.

Solution by the proposer (continued from Vol. 32, no. 1)

Proofs of parts (A) and (C) were given in the above issue of this *Quarterly*.

Proof of Part (B): We see that $U_1^{(r)}$ is a special case of $H_1^{(r)}$. Making the substitution $H_1^{(r)} \equiv U_1^{(r)}$ into part (C), the result follows at once.

Note: Although not required in the problem, we may obtain some interesting identities by taking determinants in the foregoing results. Moreover, special cases of $G_r(z)$ yield identities for the Fibonacci, Pell, Tribonacci, and Quadranacci numbers, some of which have already been studied extensively. For example, if $G_2(z) = z^2 - z - 1$, we obtain

$$U_1^{(2)} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } U_n^{(2)} = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}.$$

In the general case, $p_r(0) = |-U_1^{(r)}| = (-1)^{r^2}|U_1^{(r)}| = (-1)^r|U_1^{(r)}| = G_r(0) = -a_{r-1}$, whence the result:

$$|U_1^{(r)}| = (-1)^{r-1}a_{r-1}. \tag{**}$$

Taking determinants in parts (B) and (C), we obtain

$$|U_n^{(r)}| = (-1)^{n(r-1)}(a_{r-1})^n; \tag{***}$$

$$|H_n^{(r)}| = (-1)^{(n-1)(r-1)}(a_{r-1})^{n-1}|H_1^{(r)}|. \tag{****}$$

Of course, $(****)$ is a generalization of $(***)$. Again, special cases of $(****)$ yield some well-known results, e.g., $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$.

It is informative to apply the foregoing results for a special case which, however, has not been studied extensively. We will take $r = 3$ and $G_3(z) = z^3 - 2z^2 - z - 1$. We will choose $H_{n,j}^{(3)}$'s so that, for $j = 1, 2, 3$, $n = 0, 1, 2$, we have $U_{n,j}^{(3)} + H_{n,j}^{(3)} = 1$. We form the following brief table of values:

n	$U_{n,1}$	$U_{n,2}$	$U_{n,3}$	$H_{n,1}$	$H_{n,2}$	$H_{n,3}$
0	0	0	1	1	1	0
1	0	1	0	1	0	1
2	1	0	0	0	1	1
3	2	1	1	2	3	3
4	5	3	2	5	7	8
5	13	7	5	12	18	20
6	33	18	13	31	46	51
7	84	46	33	79	117	130
8	214	117	84	201	298	331
9	545	298	214	512	759	843
10	1388	759	545	1304	1933	2147

We omit the superscript "(3)" for brevity. Then $|U_1| = 1$ and $|H_1| = 2$, as we may verify; hence, $|H_n| = 2$ for all n. This may be left in determinant form or expanded into a sum of terms

$$(-1)^{s-1}H_{n,i}H_{n+1,j}H_{n+2,k},$$

where $i, j, k = 1, 2, 3$ in some order, and s is the minimum number of binary interchanges of digits required to obtain the triplet (i, j, k) from the initial triplet $(1, 2, 3)$. This sum then must equal 2. Clearly, many such identities may be devised.

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H-471 ***Proposed by Andrew Cusumano & Marty; Samberg, Great Neck, NY***
(Vol. 30, no. 4, November 1992)

Starting with a sequence of four ones, build a sequence of finite differences where the number of finite differences taken at each step is the term of the sequence. That is,

S_1	S_2	S_3
1 1 1 1	1 1 1 1	1 1 1 1
1 2 3 4 5	1 2 3 4 5	1 2 3 4 5
	1 2 4 7 11 16	1 2 4 7 11 16
		1 2 4 8 15 26 42

Now, reverse the procedure but start with the powers of the last row of differences and continue until differences are constant. For example, if the power is two, we have

1 4 9 16 25	1 4 16 49 121 256 etc.
3 5 7 9	3 12 33 72 135
2 2 2	9 21 39 63
	12 18 24
	6 6

The sequence of constants obtained when the power is two is 2, 6, 20, 70, ..., while the sequence of constants when the power is three is 6, 90, 1680, 34650, ...

Let N be the number of the term in the original difference sequence and M be the power used in forming the reversed sequence. Show that the constant term is

$$X(N, M) = \frac{(N \cdot M)!}{(N!)^M}, \quad N = 1, 2 3, \ldots, \quad M = 2, 3, 4, \ldots .$$

For example, $X(2, 3) = \frac{6!}{2^3} = 90$.

Solution by Paul S. Bruckman, Edmonds, WA

Let $\theta_{k,N}$ denote the k^{th} term of row S_N $(k = 1, 2, 3, \ldots)$. For example, $S_2 = (1, 2, 4, 7, 11, 16, \ldots)$ and $\theta_{4,2} = 7$. By definition, we are to have:

$$\theta_{k+1,N} - \theta_{k,N} = \theta_{k,N-1}, \tag{1}$$

$$\theta_{k,0} = 1 \text{ for all } k, \text{ and} \tag{2}$$

$$\theta_{1,N} = 1, \quad N = 1, 2, 3, \ldots . \tag{3}$$

Successive "finite integration" of (1), beginning with (2) and using (3), yields

$$\theta_{k,N} = \sum_{j=0}^{[\frac{1}{2}N]} \binom{k}{N-2j}. \tag{4}$$

We note that $\theta_{k,N}$ is a polynomial in k of degree N, whose leading term is equal to $k^N / N!$. Then $(\theta_{k,N})^M$ is a polynomial in k of degree MN, with leading term $k^{MN} / (N!)^M$. We then observe that $X(N, M) = \Delta^{MN}(\theta_{k,N})^M = \Delta^{MN}(k^{MN} / (N!)^M)$, which yields

$$X(N, M) = \frac{(MN)!}{(N!)^M}.$$

Note: *Four* "1's" in the original sequence will no longer suffice to display the constant term $X(N, M)$; the minimum number of "1's" required is $MN - N + 1$. As expected, we find that $X(N, 1) = 1$ for all N.

Also solved by M. Deshpande.

An Entry Level Job

H-472 ***Proposed by Paul S. Bruckman, Edmonds, WA***
(Vol. 30, no. 4, November 1992)

Let $Z(n)$ denote the Fibonacci entry point of the natural number n, that is, the smallest positive index t such that $n|F_t$. Prove that $n = Z(n)$ iff $n = 5^u$ or $n = 12\cdot 5^u$, for some $u \ge 0$.

Solution by the proposer

Proof: We make use of the following special values:

$$Z(2^r) = 1 \text{ if } r = 0,\ 3\cdot 2^{r-1} \text{ if } r = 1 \text{ or } 2,\ 3\cdot 2^{r-2} \text{ if } r \ge 3; \quad (1)$$

$$Z(3^s) = 1 \text{ if } s = 0,\ 4\cdot 3^{s-1} \text{ if } s \ge 1; \quad (2)$$

$$Z(5^t) = 5^t, t \ge 0. \quad (3)$$

We will also make use of the following known facts regarding Fibonacci entry points:

$$Z(p^e) = p^f Z(p), \text{ for all primes } p, \text{ where } 0 \le f < e; \quad (4)$$

$$\text{If } n = p_1^{e_1} p_2^{e_2} \cdots p_\xi^{e_\xi}, \text{ then } Z(n) = \text{LCM}[Z(p_1^{e_1}), Z(p_2^{e_2}), \ldots, Z(p_\xi^{e_\xi})]. \quad (5)$$

Since $Z(12) = 12$ and $Z(5^u) = 5^u$, we see from (5) that $Z(n) = n$ if $n = 5^u$ or $n = 12\cdot 5^u$.

Conversely, first suppose that $n = P$, where $P = 2^r 3^s$, $r, s \ge 0$. If $r \ge 3$ and $s \ge 1$, then $Z(P) = 2^{\max(r-2,\,2)} 3^{\max(s-1,\,1)}$; we see by inspecting the exponent of 2 that $z(P) = P$ is impossible. We may enumerate the remaining possibilities for r and s in the following table:

P	$Z(P)$
$2^0 3^0 = 1$	1
$2^0 3^s, s \ge 1$	$2^2 3^{s-1}$
$2^1 3^0 = 2$	3
$2^1 3^1 = 6$	12
$2^1 3^s, s \ge 2$	$2^2 3^{s-1}$
$2^2 3^0 = 4$	6
$2^2 3^1 = 12$	12
$2^2 3^s, s \ge 2$	$2^2 3^{s-1}$
$2^r 3^0, r \ge 3$	$2^{r-2} 3^1$

We see that $n = P = Z(P)$ only if $n = 1$ or 12. Moreover, if we assume that $n = P\cdot 5^u$, we see that $n = Z(n)$ only if $n = 5^u$ or $12\cdot 5^u$, $u \ge 0$.

Next, we suppose that $n = P\cdot 5^u \cdot Q$, where gcd(Q, 30) = 1 and $Q > 1$. Suppose Q has the prime factorization: $Q = \prod_{i=1}^{\omega} q_i^{e_i}$; let $q = \max(q_1, q_2, \ldots, q_\omega)$ and $q^e \| Q$. Now $Z(q^e) = q^f Z(q)$, where $0 \le f < e$ and $Z(q)$ is divisible only by primes smaller than q [since $Z(q) \le q+1$ and $\gcd(q, Z(q)) = 1$; also, $q + 1$ is even]. *A fortiori*, the same is true for the other $Z(q_i)$'s. We therefore see that $q^e \| n$ and $q^f \| Z(n)$, which shows that $n \ne Z(n)$.

This exhausts the possibilities, and the problem is solved.

Another Equivalence

H-473 ***Proposed by A. G. Schaake & J. C. Turner, Hamilton, New Zealand (Vol. 30, no. 4, November 1992)***

Show that the following (see [1], p. 98) is equivalent to Fermat's Last Theorem: "For $n > 2$ there does not exist a positive integer triple (a, b, c) such that the two rational numbers $\frac{r}{s}, \frac{p}{q}$, with

$$r = c - a. \qquad p = b - 1,$$
$$s = \sum_{i=1}^{n} b^{n-i} \qquad q = \sum_{i=1}^{n} a^{i-1}c^{n-i},$$

are penultimate and final convergents, respectively, of the simple continued fraction (having an odd number of terms) for $\frac{p}{q}$."

Reference

1. A. G. Schaake & J. C. Turner. *New Methods for Solving Quadratic Diophantine Equations (Part I and Part II)*. Research Report No. 192. Department of Mathematics and Statistics, University of Waikato, New Zealand, 1989.

Solution by Paul S. Bruckman, Edmonds, WA

Suppose there exists a positive integer triple (a, b, c) such that if p, q, r, and s are as defined in the statement of the problem then

$$\frac{r}{s} = [\theta_1, \theta_2, \ldots, \theta_{m-1}] \quad \text{and} \quad \frac{p}{q} = [\theta_1, \theta_2, \ldots, \theta_{m-1}, \theta_m], \tag{1}$$

for some sequence $\theta_1, \theta_2, \ldots, \theta_m$ of positive integers, where $m \ge 3$ is odd. The notation $[\theta_1, \theta_2, \ldots, \theta_k]$ represents the value of the simple continued fraction (s.c.f.) $= \theta_1 + 1/\theta_2 + 1/\theta_3 + \cdots + 1/\theta_k$, $k = 1, 2, \ldots, m$, also known as the k^{th} convergent of the s.c.f. for p/q.

Since r/s and p/q are supposed to be finite rationals, we require that $s > 0, q > 0$; moreover, we are interested only in positive rationals, so we require that $r > 0$ and $p > 0$. Hence, we suppose

$$b > 1, \; c > a. \tag{2}$$

Since r/s and p/q are successive convergents of a s.c.f., and since m is odd, we must have

$$rq - ps = 1. \tag{3}$$

We now note that

$$s = \frac{b^n - 1}{b - 1}, \; q = \frac{c^n - a^n}{c - a} \quad \text{(for some } n > 2\text{)}. \tag{4}$$

Then $rq - ps = c^n - a^n - (b^n - 1) = 1$, which implies

$$c^n = a^n + b^n. \tag{5}$$

Thus, our assumption implies that (a, b, c) satisfies Fermat's Last Theorem.

Conversely, suppose that there exists a positive integer triplet (a, b, c) which satisfies (5) for some $n > 2$, i.e., suppose Fermat's Last Theorem is false. If $b = a$, then $c / a = 2^{1/n}$, which is patently impossible; thus, without loss of generality, we may suppose $b < a$. Let p, q, r, and s be as defined in the statement of the problem. We seek to prove that p, q, r, and s satisfy (1) for some $m \geq 3$ odd, and some sequence $\theta_1, \theta_2, \ldots, \theta_m$ of natural numbers. We note that $rq - ps = c^n - a^n - (b^n - 1) = (c^n - a^n - b^n) + 1$, which is the statement of (3). Hence,

$$r / s - p / q = 1 / qs. \tag{6}$$

According to a well-known theorem of continued fraction theory (e.g., Theorem 184 in [1]), if $r / s - p / q < 1 / 2s^2$, then (1) holds. Therefore, in order to establish (1), it suffices to show that $1/qs < 1 / 2s^2$, or

$$q > 2s. \tag{7}$$

Now we note that

$$q = \sum_{i=1}^{n} a^{i-1} c^{n-i} > \sum_{i=1}^{n} a^{i-1} a^{n-i} = na^{n-1}.$$

Also

$$s = \sum_{i=1}^{n} b^{n-i} < \sum_{i=1}^{n} a^{n-i} = \frac{a^n - 1}{a-1} < \frac{a^n}{a-1} \quad \text{(using the assumption that } b < a\text{)}.$$

Thus,

$$q / s > \frac{n(a-1)}{a}. \tag{8}$$

Since $a > b > 1$, thus $a \geq 3$. Also, $n \geq 3$. Thus, $q / s > 3 \cdot 2 / 3$ or

$$q / s > 2. \tag{9}$$

This establishes (1).

Thus, the negative of Fermat's Last Theorem is equivalent to the negative of the statement of the problem. It follows that Fermat's Last Theorem is equivalent to the statement of the problem.

Reference

1. G. H. Hardy & E. M. Wright. *An Introduction to the Theory of Numbers*. 4th ed. Oxford: The Clarendon Press, 1960.

Also solved by the proposers.

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