AN UNEXPECTED ENCOUNTER WITH THE FIBONACCI NUMBERS

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In this article, an incident is narrated whereby the author unexpectedly came across the Fibonacci numbers while solving a problem concerning arithmetic progressions. The incident occurred when the author observed that $3+4+5+6=18=3\cdot 6$. That is, the sum of the elements in the arithmetic progression 3, 4, 5, 6 is equal to the product of the first and last terms of the progression. Its generalization can be stated as follows.

Problem: Find three positive integers a, h, and n such that

$$a+(a+h)+\cdots+(a+(n-1)h)=a(a+(n-1)h).$$

Solution: First, note that since we have an arithmetic progression, we have a solution when

$$na+\frac{n(n-1)h}{2}=a^2+a(n-1)h \tag{1}$$

which on solving for a becomes

$$a=\frac{n-(n-1)h+\sqrt{n^2+(n-1)^2h^2}}{2}. \tag{2}$$

Since a is an integer, for a solution, there must be an integer z such that $z^2=n^2+(n-1)^2h^2$ or such a triple $(n,(n-1)h,z)$ is a Pythagorean triple. Hence, by the well-known parametrization for Pythagorean triples, a solution must exist if and only if there exist integers x and y such that

$$2xy=n,\ x^2-y^2=(n-1)h,\ x^2+y^2=z, \tag{3}$$

or

$$2xy=(n-1)h,\ x^2-y^2=n,\ x^2+y^2=z. \tag{4}$$

Solving for h in both (3) and (4) and then finding the value of a in (2), we have a solution to (1) whenever there exists a pair of integers x and y such that

$$\frac{x^2-y^2}{2xy-1}=h,\ a=y(x+y),\ n=2xy \tag{5}$$

or

$$\frac{2xy}{x^2-y^2-1}=h,\ a=x(x-y),\ n=x^2-y^2. \tag{6}$$

A program in BASIC was written and run to obtain such pairs of integers x and y. A careful analysis of the output revealed that many solutions were related to the Fibonacci numbers. That is

Theorem: Let m be any positive integer, F_k be the k^{th} Fibonacci number, $n=F_{2m-1}F_{2m+2}$, $a=F_{2m}^2+1$, and $h=2$. Then, we have a solution to (1).

Proof: First, we observe the well-known facts that

$$n = F_{2m-1}F_{2m+2} = F_{2m+1}^2 - F_{2m}^2$$

and

$$a = F_{2m-1}F_{2m+1} = F_{2m}^2 + 1.$$

Now, using these identities with (1) and the definition of the Fibonacci numbers, we have

$$\begin{aligned}
n(a+(n-1)) &= (F_{2m+1}^2 - F_{2m}^2)(F_{2m}^2 + F_{2m+1}^2 - F_{2m}^2) \\
&= (F_{2m+1}^2 - F_{2m}^2)F_{2m+1}^2 \\
&= F_{2m+1}^2 F_{2m-1}F_{2m+2} \\
&= F_{2m+1}^2 F_{2m-1}(2F_{2m+1}^2 - F_{2m+1}F_{2m-1}) \\
&= F_{2m+1}F_{2m-1}(2F_{2m+1}^2 - F_{2m}^2 - 1) \\
&= F_{2m+1}F_{2m-1}(F_{2m}^2 + 1 + 2F_{2m+1}^2 - 2F_{2m}^2 - 2) \\
&= a(a+2(n-1)).
\end{aligned}$$

Hence, there exist a countable infinite set of segments of arithmetic progressions with a common difference of 2 such that the sum of the elements in the segments is equal to the product of the first and last terms. Below, we give a few examples:

$$\begin{array}{ll}
2, 4, 6 & (m=1), \\
10, 12, 14, \ldots, 40 & (m=2), \\
65, 67, 69, \ldots, 273 & (m=3), \\
442, 444, 446, \ldots, 1870 & (m=4).
\end{array}$$

The other values generated by the BASIC program did not appear to be related to the Fibonacci numbers.

The connection between the solution of the problem and elements of the Pell sequence is established by the author in an article which will appear in *Math. Student* **63** (1994).

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