ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to 72717.3515@compuserve.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-760 ***Proposed by Russell Euler, Northwest Missouri State University, Maryville, MO***

Prove that $F_{n+1}^2 \geq F_{2n}$ for all $n \geq 0$.

B-761 ***Proposed by Richard André-Jeannin, Longwy, France***

Evaluate the determinants

$$\begin{vmatrix} L_0 & L_1 & L_2 & L_3 & L_4 \\ L_1 & L_0 & L_1 & L_2 & L_3 \\ L_2 & L_1 & L_0 & L_1 & L_2 \\ L_3 & L_2 & L_1 & L_0 & L_1 \\ L_4 & L_3 & L_2 & L_1 & L_0 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} L_0^2 & L_1^2 & L_2^2 & L_3^2 & L_4^2 \\ L_1^2 & L_0^2 & L_1^2 & L_2^2 & L_3^2 \\ L_2^2 & L_1^2 & L_0^2 & L_1^2 & L_2^2 \\ L_3^2 & L_2^2 & L_1^2 & L_0^2 & L_1^2 \\ L_4^2 & L_3^2 & L_2^2 & L_1^2 & L_0^2 \end{vmatrix}.$$

B-762 ***Proposed by Larry Taylor, Rego Park, NY***

Let n be an integer.

(a) Generalize the numbers (2, 2, 2) to form three three-term arithmetic progressions of integral multiples of Fibonacci and/or Lucas numbers with common differences $3F_n$, $5F_n$, and $3F_n$.

(b) Generalize the numbers (4, 4, 4) to form two such arithmetic progressions with common differences F_n and F_n.

(c) Generalize the numbers (6, 6, 6) to form four such arithmetic progressions with common differences F_n, $5F_n$, $7F_n$, and F_n.

B-763 *Proposed by Juan Pla, Paris, France*

Let $A = \begin{pmatrix} e^{i\pi/3} & \sqrt{2} \\ \sqrt{2} & e^{-i\pi/3} \end{pmatrix}$. Express A^n in terms of Fibonacci and/or Lucas numbers.

B-764 *Proposed by Mark Bowron, Tucson, AZ*

Consider row n of Pascal's triangle, where n is a fixed positive integer. Let S_k denote the sum of every fifth entry, beginning with the k^{th} entry, $\binom{n}{k}$. If $0 \le i < j < 5$, show that $|S_i - S_j|$ is always a Fibonacci number.

For example, row 10 of Pascal's triangle is $1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1$. Thus, $S_0 = 1 + 252 + 1 = 254$, $S_1 = 10 + 210 = 220$, and $254 - 220 = 34 = F_9$.

B-765 *Proposed by Mohammad K. Azarian, University of Evansville, Evansville, IN*

Let m and n be positive integers greater than 1, and let $x = F_{mn} / (F_m F_n)$. What famous constant is represented by

$$\left[\sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}\frac{1}{j!}\right)\left(\frac{1}{x^i} - \frac{1}{x^{i+1}}\right)\right]^x ?$$

SOLUTIONS

The Determination

B-731 *Proposed by H.-J. Seiffert, Berlin, Germany* *(Vol. 31, no. 1, February 1993)*

Evaluate the determinant:

$$\begin{vmatrix} F_0 & F_1 & F_2 & F_3 & F_4 \\ F_1 & F_0 & F_1 & F_2 & F_3 \\ F_2 & F_1 & F_0 & F_1 & F_2 \\ F_3 & F_2 & F_1 & F_0 & F_1 \\ F_4 & F_3 & F_2 & F_1 & F_0 \end{vmatrix}.$$

Generalize.

Solution 1 by Leonard A. G. Dresel, Reading, England

Let M_n denote the $n \times n$ matrix (m_{ij}) where $m_{ij} = F_{|i-j|}$ so that the given determinant is $\det(M_5)$. For $n \ge 3$, $\det(M_n)$ remains unchanged if we subtract the sum of the second and third rows from the first row. This modified first row becomes $(-2, 0, 0, \ldots)$ because $F_0 - F_1 - F_2 = -2$, $F_1 - F_0 - F_1 = 0$, and the remaining elements in the first row vanish because of the recurrence for F_j.

Hence, for $n \ge 3$, we have $\det(M_n) = (-2)\det(M_{n-1})$. But, for $n = 2$, we have $\det(M_2) = F_0^2 - F_1^2 = -1$. Hence, we have by induction that, for $n \ge 2$, $\det(M_n) = -(-2)^{n-2}$. In particular, $\det(M_5) = 8$.

Solution 2 by Pentti Haukkanen, University of Tampere, Tampere, Finland

Replace the Fibonacci sequence (F_n) by a sequence (w_n) that satisfies $w_n = pw_{n-1} + w_{n-2}$ $(n \geq 2)$ with initial condition $w_0 = 0$, where p and w_1 are arbitrary constants. Then, as in solution 1, subtract p times the second row plus the third row from row 1. We find that the first row becomes $(-pw_1 - w_2, 0, 0, \ldots)$. By induction, the value of the $n \times n$ determinant is $(-pw_1 - w_2)^{n-2}(-w_1^2)$. Since $w_2 = pw_1$, this can be written as $(-1)^{n-1}(2p)^{n-2}w_1^n$.

The proposer submitted the general case for an $n \times n$ matrix. This generalization was found by all solvers. In addition, Suck found the same generalization as Huakkanen. Bruckman obtained the result for a generalized Fibonacci sequence with two arbitrary initial conditions, but the result is a bit messy. See problem B-761 in this issue for a related problem.

Also solved by Richard André-Jeannin, Seung-Jin Bang, Glenn Bookhout, Scott H. Brown, Paul S. Bruckman, the Con Amore Problem Group, Russell Jay Hendel, Ed Korntved, Harris Kwong, Carl Libis, Graham Lord, Bob Prielipp, Sahib Singh, J. Suck, Ralph Thomas, A. N. 't Woord, and the proposer.

The Mod Squad

B-732 *Proposed by Richard André-Jeannin, Longwy, France (Vol. 31, no. 1, February 1993)*

Let (w_n) be any sequence of integers that satisfies the recurrence $w_n = pw_{n-1} - qw_{n-2}$ where p and q are odd integers. Prove that, for all n, $w_{n+6} \equiv w_n \pmod 4$.

Solution by Ed Korntved, Morehead, KY

Since p and q are odd integers, both are congruent to ± 1 modulo 4. The sequence is completely determined by the first two elements of the sequence. There are four cases.

Case 1: $p \equiv 1, q \equiv 1 \pmod 4$. The recurrence becomes $w_n \equiv w_{n-1} - w_{n-2} \pmod 4$. Starting with $w_1 = a$ and $w_2 = b$, the sequence, modulo 4, would be $a, b, b-a, -a, -b, -b+a, a, b, \ldots$, which is easily seen to have a period of 6 modulo 4.

Case 2: $p \equiv 1, q \equiv -1 \pmod 4$. The recurrence becomes $w_n \equiv w_{n-1} + w_{n-2} \pmod 4$. The sequence would now be $a, b, a+b, a+2b, 2a+3b, 3a+b, a, b, \ldots$, which also has a period of 6 modulo 4.

Case 3: $p \equiv -1, q \equiv -1 \pmod 4$ yields the sequence $a, b, -b+a, 2b-a, -3b+2a, b-3a, a, b, \ldots$, which has period 6.

Case 4: $p \equiv -1, q \equiv 1 \pmod 4$ yields the sequence $a, b, -a-b, a, b, -a-b, a, b, \ldots$, which also repeats every 6 terms.

Seiffert showed that $w_{n+12} \equiv w_n \pmod 8$. Is this the beginning of a trend?

Also solved by Seung-Jin Bang, Paul S. Bruckman, Joseph E. Chance, the Con Amore Problem Group, Charles K. Cook, Leonard A. G. Dresel, Herta T. Freitag, Jane E. Friedman, Russell Jay Hendel, Harris Kwong, Carl Libis, Don Redmond, H.-J. Seiffert, Sahib Singh, Lawrence Somer, J. Suck, Ralph Thomas, A. N. 't Woord, and the proposer. One incorrect solution was received.

B-733 *Proposed by Piero Filipponi, Rome, Italy (Vol. 31, no. 1, February 1993)*

Write down the Pell sequence, defined by $P_0 = 0$, $P_1 = 1$, and $P_{n+2} = 2P_{n+1} + P_n$ for $n \geq 0$. Form a difference triangle by writing down the successive differences in rows below it. For example,

$$
\begin{array}{cccccccccccccccc}
0 & & 1 & & 2 & & 5 & & 12 & & 29 & & 70 & & 169 & \cdots \\
 & 1 & & 1 & & 3 & & 7 & & 17 & & 41 & & 99 & & \\
 & & 0 & & 2 & & 4 & & 10 & & 24 & & 58 & & & \\
 & & & 2 & & 2 & & 6 & & 14 & & 34 & & & & \\
 & & & & 0 & & 4 & & 8 & & 20 & & & & & \\
 & & & & & 4 & & 4 & & 12 & & & & & & \\
 & & & & & & 0 & & 8 & & & & & & & \\
 & & & & & & & 8 & & & & & & & & \\
 & & & & & & & & \ddots & & & & & & &
\end{array}
$$

Identify the pattern that emerges down the left side and prove that this pattern continues.

Solution by Russell Jay Hendel, Morris College, Sumter, SC

It is straightforward to show that, if any row satisfies a linear recurrence with constant coefficients, then the difference row below it also satisfies the same recurrence. Thus, each row of Pell's difference triangle satisfies the Pell recurrence.

Now if a and b are any two successive terms in some row, then we have the following subtriangle:

$$
\begin{array}{ccccc}
a & & b & & 2b+a \\
 & b-a & & b+a & \\
 & & 2a & &
\end{array}
$$

Thus, any element is twice the one two rows back (along the diagonal). Since the leftmost diagonal begins with a 0 and then a 1, it follows that every second element along the diagonal is a 0 and that the intervening elements are successive powers of 2.

Luchins and Hendel have found the pattern down the leftmost diagonal for the difference triangle of an arbitrary linear recurrence with constant coefficients. Their result is announced in [1]. See the previous issue of this Quarterly for more fun with Pell numbers.

Reference

1. Edith H. Luchins & Russell J. Hendel. Abstract 883-11-193: "Operators that Take Sequences to Diagonals of Their Difference Triangles." *Abstracts of the American Mathematical Society* **14** (1993):461.

Also solved by Richard André-Jeannin, Seung-Jin Bang, Paul S. Bruckman, Joseph E. Chance, the Con Amore Problem Group, Leonard A. G. Dresel, Russell Euler, Herta T. Freitag, Harris Kwong, H.-J. Seiffert, Tony Shannon, Sahib Singh, J. Suck, Ralph Thomas, David Tuller, A. N. 't Woord, and the proposer.

Powers of 5

B-734 ***Proposed by Paul S. Bruckman, Edmonds, WA (Vol. 31, no. 1, February 1993)***

If r is a positive integer, prove that $L_{5^r} \equiv L_{5^{r-1}} \pmod{5^r}$.

Solution 1 by Leonard A. G. Dresel, Reading, England

From identity #83 of [3], we have

$$F_{5n} = F_n\left[25F_n^4 + 25(-1)^n F_n^2 + 5\right]. \tag{1}$$

Similarly, it is straightforward to show that

$$L_{5n} = L_n\left[25F_n^4 + 15(-1)^n F_n^2 + 1\right]. \tag{2}$$

From equations (1) and (2), we see that

$$F_{5n} \equiv 0 \pmod{5F_n} \tag{3}$$

and

$$L_{5n} \equiv L_n \pmod{5F_n^2}. \tag{4}$$

Equation (3) can be written as $5F_n \mid F_{5n}$. Since $F_1 = 1$, it follows by induction that

$$5^r \mid F_{5^r}. \tag{5}$$

From equation (4) with $n = 5^{r-1}$, we have $5F_n^2 \mid (L_{5n} - L_n)$, so $5(5^{r-1})^2 \mid (L_{5n} - L_n)$ or $L_{5^r} \equiv L_{5^{r-1}}$ (mod 5^{2r-1}), which generalizes the proposer's problem.

Singh, Somer, Suck, and Woord also found this generalization. Singh notes that since L_{5^r} is always odd [this follows from the identity $L_{5n} = L_n^5 - 5(-1)^n L_n^3 + 5L_n$], we have the even stronger congruence: $L_{5^r} \equiv L_{5^{r-1}} \pmod{2 \cdot 5^{2r-1}}$. Prielipp points out that property (5) is given in [1]. Singh found it in [2]. Somer found it in [4].

References

1. V. E. Hoggatt, Jr. Problem B-248. *The Fibonacci Quarterly* **11** (1973):553.
2. Verner E. Hoggatt, Jr., & Gerald E. Bergum. "Divisibility and Congruence Relations." *The Fibonacci Quarterly* **12** (1974):189-95.
3. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section: Theory and Applications.* Chichester: Ellis Horwood Ltd., 1989.
4. D. D. Wall. "Fibonacci Series Modulo *m*." *Amer. Math. Monthly* **67** (1960):525-32.

Solution 2 by Don Redmond, Southern Illinois University, Carbondale, IL

Let (V_n) be a sequence defined by $V_{n+2} = aV_{n+1} + V_n$ with initial conditions $V_0 = 2$ and $V_1 = a$, where a is a positive integer. If p is an odd prime and r is a positive integer, we will show that

$$V_{p^r} \equiv V_{p^{r-1}} \pmod{p^r}. \tag{1}$$

In particular, if $a = 1$, then V_n becomes L_n and we have

$$L_{p^r} \equiv L_{p^{r-1}} \pmod{p^r}. \tag{2}$$

This generalizes the current proposal, for which $p = 5$.

To prove (1), we will use the fact that, if n is odd, then

$$V_{pn} = V_n^p + \sum_{k=1}^{(p-1)/2} pb_k V_n^{p-2k}, \tag{3}$$

where $b_k = \frac{1}{k} \cdot \binom{p-k-1}{k-1}$. This was proven by Lucas in 1878 (see [2], p. 38). Since $c_k = pb_k$ can also be written as $\binom{p-k-1}{k} + 2\binom{p-k-1}{k-1}$, it is clear that c_k is an integer. Furthermore, since p is prime and $k < p$, we see that c_k is divisible by p, so b_k is an integer for all k.

We will proceed to prove equation (1) by induction. If equation (1) is true for some r, then we would have $V_{p^r} = V_{p^{r-1}} + p^r T$ for some integer T. Then by identity (3) with $n = p^r$ we find

$$V_{p^{r+1}} = (V_{p^{r-1}} + p^r T)^p + \sum_{k=1}^{(p-1)/2} pb_k (V_{p^{r-1}} + p^r T)^{p-2k}$$
$$= V_{p^{r-1}}^p + \sum_{k=1}^{(p-1)/2} pb_k V_{p^{r-1}}^{p-2k} + \text{ a multiple of } p^{r+1}$$
$$= V_{p^r} + \text{ a multiple of } p^{r+1},$$

which shows that equation (1) is true for $r+1$.

But equation (1) is true for $r=1$, since $V_p \equiv V_1^p = a^p \pmod{p}$ follows from identity (3) when $n=1$; and $a^p \equiv a \pmod{p}$ is true by Fermat's Little Theorem, since p is a prime.

Thus, the induction is complete.

André-Jeannin also proved equation (2) and Seiffert found equation (2) in [1], p. 111.

References

1. D. Jarden. *Recurring Sequences*, 3rd ed. Jerusalem: Riveon Lematematika, 1973.
2. Edouard Lucas. *The Theory of Simply Periodic Numerical Functions*. Santa Clara, Calif.: The Fibonacci Association, 1969.

Also solved by Richard André-Jeannin, Seung-Jin Bang, the Con Amore Problem Group, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, J. Suck, Ralph Thomas, A. N.'t Woord, and the proposer.

Square Root of a Recurrence

B-735 ***Proposed by Curtis Cooper and Robert E. Kennedy, Central Missouri State Asylum for Crazed Mathematicians, Warrensburg, MO***
(Vol. 31, no. 1, February 1993)

Let the sequence (y_n) be defined by the recurrence

$$y_{n+1} = 8y_n + 22y_{n-1} - 190y_{n-2} + 28y_{n-3} + 987y_{n-4} - 700y_{n-5} - 1652y_{n-6} + 1652y_{n-7}$$
$$+700y_{n-8} - 987y_{n-9} - 28y_{n-10} + 190y_{n-11} - 22y_{n-12} - 8y_{n-13} + y_{n-14}$$

for $n \geq 15$ with initial conditions given by the table:

n	y_n
1	1
2	1
3	25
4	121
5	1296
6	9025
7	78961
8	609961
9	5040025
10	40144896
11	326199721
12	2621952025
13	21199651201
14	170859049201
15	1379450250000

Prove that y_n is a perfect square for all positive integers n.

Editorial Note: *For many years, back almost to the dawn of time, Paul S. Bruckman has been solving every single problem proposed in this column. When this "insane" problem came in, I jumped with the thought: "Aha! Now I can stump Bruckman." The proposers' solution involves pulling the recurrence* $x_{n+1} = x_n + 5x_{n-1} + x_{n-2} - x_{n-3}$ *out of a hat. It is then not too hard to show that the squares of the elements of this recurrence satisfy the original recurrence. My feeling was that there was no way anyone could find this "rabbit," and yet the proposers' solution was so simple that I could claim this problem was suitable for the Elementary Problem Column. Anyway, less than a week after the journal hit the newsstands, much to my chagrin, I received a letter from Bruckman containing a solution!*

Several other readers also pulled the same rabbit out of the hat. They must be commended.

Solution 1 by the proposers

Let $x_{n+1} = x_n + 5x_{n-1} + x_{n-2} - x_{n-3}$, for $n \geq 4$ with initial conditions $x_1 = x_2 = 1, x_3 = 5$, and $x_4 = 11$.

We will show, by induction on n, that $y_n = x_n^2$ for $n \geq 1$.

The result is numerically true for $n = 1, 2, \ldots, 15$. Suppose the result is true for all $k < n$ where $n \geq 16$. Then, by the induction hypothesis,

$$\begin{aligned} y_{n+1} - x_{n+1}^2 &= 8x_n^2 + 22x_{n-1}^2 - 190x_{n-2}^2 + 28x_{n-3}^2 + 987x_{n-4}^2 - 700x_{n-5}^2 - 1652x_{n-6}^2 \\ &\quad + 1652x_{n-7}^2 + 700x_{n-8}^2 - 987x_{n-9}^2 - 28x_{n-10}^2 + 190_{n-11}^2 - 22x_{n-12}^2 \\ &\quad - 8x_{n-13}^2 + x_{n-14}^2 - (x_n + 5x_{n-1} + x_{n-2} - x_{n-3})^2. \end{aligned}$$

In the right-hand side, make the substitutions $x_k = x_{k-1} + 5x_{k-2} + x_{k-3} - x_{k-4}$, for $k = n, n-1, \ldots, n-10$. A mere few hours of algebraic simplification then reveals that the right-hand side is identically 0. Thus, $y_{n+1} = x_{n+1}^2$ and the induction is complete.

All other solvers found that y_n satisfies the simpler recurrence,

$$y_{n+1} = 5y_n + 35y_{n-1} - 67y_{n-2} - 145y_{n-3} + 145y_{n-4} + 67y_{n-5} - 35y_{n-6} - 5y_{n-7} + y_{n-8}.$$

One of their solutions will be printed in a future issue, if space permits.

Also solved by Paul S. Bruckman, the Con Amore Problem Group, Leonard A. G. Dresel, H.-J. Seiffert, and A. N. 't Woord.

❖❖❖