# FIBONACCI NUMBERS AND A CHAOTIC PIECEWISE LINEAR FUNCTION

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## INTRODUCTION

The continuous piecewise linear function defined by

$$g(x)=\begin{cases} x+1/2 & \text{for } x \text{ in } H=[0,1/2] \\ 2(1-x) & \text{for } x \text{ in } I=[1/2,1] \end{cases}$$

was displayed by Xun Cheng Huang in [1, p. 97] as an example of a function having periodic points of every finite order $n$ under iteration by $g$ where $g^n(x)=g(g^{n-1}(x))$, with $g^0(x)=x$. We shall examine the iterates of $g$, and show that there are $F_{n+2}$ subintervals of $U=[0,1]$ on which $g^n$ is linear, of which $F_n$ lie in $H$ and $F_{n+1}$ lie in $I$. Of the $F_n$ intervals in $H$, $F_{n-2}$ are mapped by $g^n$ onto $I$ and $F_{n-1}$ onto $U$; of the $F_{n+1}$ in $I$, $F_{n-1}$ are mapped onto $I$ and $F_n$ onto $U$; by $g^n$. Furthermore, the number of points in $U$ whose period is a factor of $n$ under iteration by $g$ is the Lucas number $L_n=F_{n-1}+F_{n+1}$. Finally, we examine the cycles in which rational numbers in $U$ with any given odd denominator appear under iteration by $g$.

## BUNS AND BINS, BUNKS AND BINKS

We shall call an interval mapped bijectively on $U$ by $g^n$ a "bun" and an interval mapped bijectively on $I=[1/2,1]$—but not in a bun—a "bin." "Bunks and "binks" are buns and bins of a fixed width $2^{-k}$. Each of the $F_{n+1}$ buns in $U$ and $F_{n-1}$ bins in $I$ contain a periodic point $x$ such that $g^n(x)=x$, so there are $L_n$ points $x$ in $U$ whose period under $g$ is a factor of $n$.

We denote by $H_{m,k}$ or $I_{m,k}$ a bin or a bun of width $2^{-k}$ having one endpoint $x=m/2^k$ such that $g^n(x)=1$. If $m$ is odd, the bink $H_{m,k}$ is adjacent to the bunk $I_{m,k}$, preceding it for odd $k$, or following it for even $k$. There are $F_n$ such pairs. There are no bins with even $m$. Of the $F_{n-1}$ buns with even $m$ one is $I_{0,k}$ if $2n=3k+1$, or one is $I_{m,k}$, $m=2^k$, if $2n=3k$. The remaining buns are adjacent pairs, one twice as wide as the other, such as $I_{12,4}$ and $I_{6,3}$ for $n=5$ that have the common endpoint $x=12/16=6/8$.

When $n=1$, the $F_3=2$ intervals are $H=H_{1,1}=[0,1/2]$; and $I=I_{1,1}=[1/2,1]$. The $F_{n+2}=3$, 5, and 8 intervals for $n=$ 2, 3, and 4 are

| | |
|---|---|
| $I_{01};\ I_{32}\ H_{32}$ | $n=2, F_4=3$ |
| $I_{12}\ H_{12};\ H_{53}\ I_{53},\ I_{42}$ | $n=3, F_5=5$ |
| $H_{13}\ I_{13}\ I_{22};\ I_{43}, I_{11,4}\ H_{11,4}, H_{73}\ I_{73}$ | $n=4, F_6=8$ |

We separate by a semicolon the buns and bins in $H$ from those in $I$. To proceed from one $n$ to the next, we first list the intervals in $I$ for $n-1$ as the intervals in $H$ for $n$ with the same $k$, but with $m$ replaced by $m-2^{k-1}$ (or $x$ by $x-1/2$). Then, after a semicolon, we list all the intervals for $n-1$

in reverse order as intervals in $I$ for $n$, but with $k$ replaced by $k+1$ and $m$ by $2^{k+1}-m$, thus replacing $x = m/2k$ by $y = 1-x/2$, since $g$ replaces $y > 1/2$ by $g(y) = 2(1-y) = x$.

We assume as induction hypothesis that, for $n = N-1$ there are $F_{n-2}$ bins and $F_{n-1}$ buns in $H$, and $F_{n-1}$ bins and $F_n$ buns in $I$, for a total of $F_n$ bins and $F_{n+1}$ buns in $U$, of which $F_n$ intervals are in $H$ and $F_{n+1}$ in $I$. We verify this for $n = 2$ and 3. Then, since $F_{n-1}+F_n = F_{n+1}$, the construction given above shows that the same is true for $n = N$, proving the hypothesis for all $n > 2$.

For $n = 5$ we list the $F_7 = 13$ buns and bins as follows:

$$I_{0,3},\ I_{3,4}\, H_{3,4},\ H_{3,3}\, I_{3,3};\ I_{9,4}\, H_{9,4},\ H_{21,5}\, I_{21,5},\ I_{12,4}\, I_{6,3},\ I_{15,4}\, H_{15,4}$$

Next we classify the binks and bunks for fixed $n$ and $k$, and count them using binomial coefficients $b_{n,k}$ defined by

$$b_{n,k} = \binom{k-1}{n-k} = \binom{k-1}{2k-n-1} = \binom{k}{n-k} - \binom{k-1}{n-k-1} = b_{n+1,k+1} - b_{n-1,k}$$

assuming $0 \le n-k \le k$. The sum over $k$ of $b_{n,k}$ is $F_n$.

For $2 < n \ge 2k$ the distributions are found to be as follows:

| | In $H$ | In $I$ | In $U$ | |
|---|---|---|---|---|
| Binks ($m$ odd) | $b_{n-2,k-1}$ | $b_{n-1,k-1}$ | $b_{n,k}$ | $n>1$ |
| Bunks ($m$ odd) | $b_{n-2,k-1}$ | $b_{n-1,k-1}$ | $b_{n,k}$ | $n>1$ |
| Bunks ($m$ even) | $b_{n-3,k-1}$ | $b_{n-2,k-1}$ | $b_{n-1,k}$ | $n>2$ |
| Bunks (all $m$) | $b_{n-1,k}$ | $b_{n,k}$ | $b_{n+1,k+1}$ | |

Summing over $k$, we replace $b_{n-i,k-j}$ by $F_{n-i}$, since

$$\Sigma_k\, b_{n,k} = \Sigma_k \binom{k-1}{n-k} = F_n.$$

For $n > 2$ we prove this count by induction, first checking its validity for $n = 3$ and 4. Bink and bunk counts for $g^n$ in $H$ are those for $g^{n-1}$ in $I$, with $n$ replaced by $n-1$. Bink and bunk counts for $g^n$ in $I$ are those for $g^{n-1}$ in $U$, with $n$ and $k$ replaced by $n-1$ and $k-1$, since $g$ doubles widths of intervals in $I$ mapped on $U$. Thus, the counts are valid for $n > 2$.

## PERIODIC POINTS

A periodic point $x$ such that $g^n(x) = x$ is contained in each of the $F_{n+1}$ intervals $I_{m,k}$ for $g^n$ that map onto $U$, but only in the $F_{n-1}$ intervals $H_{m,k}$ in $I$, since $g^n$ maps $H_{m,k}$ intervals in $H$ onto $I$ without overlap. Thus, the number of periodic points in $U$ whose periods divide $n$ is

$$L_n = F_{n-1} + F_{n+1} = \tau^n + (-\tau)^{-n}, \text{ where } \tau = (5^{1/2}+1)/2.$$

The coordinate $x$ of the periodic point in an $I_{m,k}$ interval is

$$x = (m+(-1)^k(x-1))/2^k = (m-(-1)^k)/(2^k-(-1)^k).$$

The coordinate $x$ of the periodic point in an $H_{m,k}$ interval is

$$x = 1-y/2 = (m+(-1)^k y)/2^k = (m+2(-1)^k)/(2^k+2(-1)^k).$$

For $n = 5$ the 11 intervals and periodic points for $g$ are

$$I_{0,3},\ I_{3,4}\ I_{3,3};\ I_{9,4}\ H_{9,4},\ H_{21,5}\ I_{21,5},\ I_{12,4}\ I_{6,3},\ I_{15,4}\ H_{15,4}$$

Note that $H_{m,k}$ intervals yield even denominators. The point $22/33 = 2/3$ with $k = n$ is a fixed point of $g$. The others form two period 5 cycles with $k = 3$ and 4, respectively:

$$(1/9, 11/18, 7/9, 4/9, 17/18),\ (2/15, 19/30, 11/15, 8/15, 14/15).$$

Each of the $\phi(b)$ rational numbers $x = a/b$ in $U$ with $b$ odd and $(a, b) = 1$ is periodic under iterations of $g$. For $x$ in $I$ we have $bg(x) = 2b(1-a/b) \equiv -2a$ (mod $b$), whereas for $x$ in $H$ we have $g(x) = a/b + 1/2, bg^2(x) = 2b(1/2 - a/b) \equiv -2a$ (mod $b$). If $t$ is the exponent of $-2$ (mod $b$), there are $t$ fractions $j/b$ in the cycle with $a/b$, such that $0 < j < b$. These $j$ form a coset of the subgroup generated by $b-2$ in the group $\phi(b)$ residues relatively prime to $b$. If $-2$ is a quadratic residue of $b$, then $t$ divides $\phi(b)/2$. If $h$ is the number of the $j/b$ in the cycle with $a/b$ that lie in $H$, then the cycle contains $h$ fractions with denominator $2b$, and has length $n = t + h$. The cycle containing $1 - a/b$ has $t - h$ denominators $2b$ and length $2t - h$. There are a total of $\phi(b)/t$ cycles containing the $\phi(b)$ fractions $j/b$ and $\phi(b)/2$ fractions $(2j+b)/2b < 1$.

To illustrate the theory, we give some examples:

**(a)** If $b = 23$, -2 is a quadratic nonresidue (mod $b$), to $t = 22$ and $h = 11$. Since $23 \times 89 = 2^{11} - 1$, 23 divdides $2^t - (-1)^t$.

**(b)** If $b = 19$, $-2 = 6^2$ (mod $b$, so $t$ divides 9. Powers of $-2$ (mod 19) are congruent to -2, 4, $-8$, $-3$, 6, 7, 5, 9, 1, so $h = 6$ of these nine are between 0 and 19/2. Thus, 1/19 and 18/19 are in cycles of $n = 9 + 6$ and $9 + 3$. Since $513 = 27 \times 19$, $b$ divides $2^9 + 1$.

**(c)** If $b = 33$, the powers of $-2$ (mod 33) are $-2$, 4, $-8$, 16, 1, so $(t, h) = (5, 3)$ and (5, 2) for cycles with $a/b$ = 1/33 and 32/33. Since $\phi(b) = 20$, there are two other cycles like these.

## REFERENCE

1. Xun Cheng Huang. "From Intermediate Value to Chaos." *Math. Magazine* **65.2** (1992):97.

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