A BRACKET FUNCTION TRANSFORM AND ITS INVERSE

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The object of this paper is to present a bracket function transform together with its inverse and some applications. The transform is the analogue of the binomial coefficient transform discussed in [2]. The inverse form will be used to give a short proof of an explicit formula in [1] for $R_k(n)$, the number of compositions of n into exactly k relatively prime summands.

Theorem 1—Bracket Function Transform: Define

$$S(n)=\sum_{k=1}^{n}\left[\frac{n}{k}\right]A_k=\sum_{j=1}^{n}\sum_{d\mid j}A_d, \tag{1}$$

$$\mathcal{A}(x)=\sum_{n=1}^{\infty}x^nA_n, \tag{2}$$

and

$$\mathcal{S}(x)=\sum_{n=1}^{\infty}x^nS_n. \tag{3}$$

Then

$$\mathcal{S}(x)=\frac{1}{1-x}\sum_{n=1}^{\infty}A_n\frac{x^n}{1-x^n}. \tag{4}$$

Proof: We need the fact that

$$\sum_{n=k}^{\infty}\left[\frac{n}{k}\right]x^{n-k}=\frac{1}{(1-x)(1-x^k)},\ k\geq 1,\ |x|<1, \tag{5}$$

which is easily proved and is the bracket function analogue of the binomial series

$$\sum_{n=k}^{\infty}\binom{n}{k}x^{n-k}=\frac{1}{(1-x)(1-x)^k},\ k\geq 1,\ |x|<1. \tag{6}$$

Relations (5) and (6) were exhibited and applied in [1] for the purpose of establishing some number theoretic congruences.

By means of (5) we may obtain the proof of (4) as follows:

$$\sum_{n=1}^{\infty}x^n\sum_{k=1}^{n}\left[\frac{n}{k}\right]A_k=\sum_{k=1}^{\infty}A_k\sum_{n=k}^{\infty}\left[\frac{n}{k}\right]x^n$$
$$=\sum_{k=1}^{\infty}A_kx^k\sum_{n=k}^{\infty}\left[\frac{n}{k}\right]x^{n-k}=\sum_{k=1}^{\infty}x^kA_k\frac{1}{(1-x)(1-x^k)},$$

which completes the proof.

Note that (4) does not turn out as nicely as the corresponding result in [2] because we now have $1-x^k$ instead of $(1-x)^k$, which is the striking difference between (5) and (6). As a result, we are not able to express $\mathcal{S}(x)$ as some function multiplied times $\mathcal{A}(x)$ as we did in [2]. Nevertheless, the result does express $\mathcal{S}$ in terms of A instead of S.

Transform (1) may next be inverted by use of the Möbius inversion theorem, but this requires some care. Here is how we do it:

$$S(n)-S(n-1)=\sum_{k=1}^{n}\left[\frac{n}{k}\right]A_k-\sum_{k=1}^{n-1}\left[\frac{n-1}{k}\right]A_k,$$

or just

$$S(n)-S(n-1)=\sum_{k=1}^{n}\left\{\left[\frac{n}{k}\right]-\left[\frac{n-1}{k}\right]\right\}A_k. \tag{7}$$

However,

$$\left[\frac{n}{k}\right]-\left[\frac{n-1}{k}\right]=\begin{cases}1 & \text{if } k|n,\\ 0 & \text{if } k\nmid n,\end{cases}$$

so that we find the relation

$$S(n)-S(n-1)=\sum_{d|n}A_d, \tag{8}$$

which may be inverted at once by the standard Möbius theorem to get

$$A(n)=\sum_{d|n}\mu\left(\frac{n}{d}\right)\{S(d)-S(d-1)\}. \tag{9}$$

It is easy to see that the steps may be reversed and we may, therefore, enunciate the bracket function inversion pair as

Theorem 2—Bracket Function Inverse Pair:

$$S(n)=\sum_{k=1}^{n}\left[\frac{n}{k}\right]A_k=\sum_{j=1}^{n}\sum_{d|j}A_d \tag{10}$$

if and only if

$$A(n)=\sum_{d|n}\mu\left(\frac{n}{d}\right)\{S(d)-S(d-1)\}. \tag{11}$$

This inversion pair is the dual of the familiar binomial coefficient pair

$$S(n)=\sum_{k=0}^{n}\binom{n}{k}A_k \tag{12}$$

if and only if

$$A_n=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}S(k). \tag{13}$$

Sometimes it will be convenient to restate the pair (10)-(11) as

Theorem 3:

$$f(n,k)=\sum_{j=1}^{n}\left[\frac{n}{j}\right]g(j,k)=\sum_{j=1}^{n}\sum_{d|j}g(d,k)$$

if and only if

$$g(n,k)=\sum_{d|n}\mu\left(\frac{n}{d}\right)\{f(d,k)-f(d-1,k)\}. \tag{15}$$

We will apply this form of our inversion theorem to give a short proof of a formula in [1]. In that paper the expansion

$$\binom{n}{k}=\sum_{j=1}^{n}\left[\frac{n}{j}\right]R_k(j)=\sum_{j=1}^{n}\sum_{\substack{d|j\\ d\ge k}}R_k(d) \tag{16}$$

was first proved, where $R_k(n)$ = the number of compositions of n into exactly k relatively prime positive summands, i.e., the number of solutions of the Diophantine equation $n=a_1+a_2+a_3+\cdots+a_k$ where $1\le a_i\le n$ and $(a_1,a_2,a_3,\dots,a_k)=1$.

Applying (14)-(15) to this, we obtain

$$R_k(n)=\sum_{d|n}\mu\left(\frac{n}{d}\right)\left\{\binom{d}{k}-\binom{d-1}{k}\right\}=\sum_{d|n}\mu\left(\frac{n}{d}\right)\binom{d-1}{k-1},$$

which proves the desired formula for $R_k(n)$.

The series (11) may be restated in the form

$$A_n=\sum_{k=1}^{n}H_k^n S(k), \tag{17}$$

but it is awkward to give a succinct expression for the H_k^n coefficients. To obtain these numbers, however, we may proceed as follows. From (11), we have

$$\begin{aligned}A(n)&=\sum_{d|n}\mu\left(\frac{n}{d}\right)S(d)-\sum_{d|n}\mu\left(\frac{n}{d}\right)S(d-1)=\sum_{d|n}\mu\left(\frac{n}{d}\right)S(d)-\sum_{(d+1)|n}\mu\left(\frac{n}{d+1}\right)S(d)\\&=\sum_{k=1}^{n}\left\{\left[\frac{n}{d}\right]-\left[\frac{n-1}{k}\right]\right\}\mu\left(\frac{n}{k}\right)S(k)-\sum_{k=1}^{n}\left\{\left[\frac{n}{k+1}\right]-\left[\frac{n-1}{k+1}\right]\right\}\mu\left(\frac{n}{k+1}\right)S(k)-\mu(n)S(0)\end{aligned}$$

so that we have the following explicit formula for the H coefficients:

$$H_k^n=\left\{\left[\frac{n}{k}\right]-\left[\frac{n-1}{k}\right]\right\}\mu\left(\frac{n}{k}\right)-\left\{\left[\frac{n}{k+1}\right]-\left[\frac{n-1}{k+1}\right]\right\}\mu\left(\frac{n}{k+1}\right)\quad\text{for } 1\le k\le n. \tag{18}$$

Ordinarily, $S(0)$ from (1) has the value 0; however, it is often convenient to modify (1) and define

$$S(n)=1+\sum_{k=1}^{n}\left[\frac{n}{k}\right]A_k \tag{19}$$

so that $S(0)=1$. With this train of thought in mind, we present a table of H_k^n for $0\le k\le n$, $n=0(1)18$, so that the table may be used for either situation. Thus, the 0-column in the array will be given by $-\mu(n)$, but with $H_0^0=1$.

A way to check the rows in the table of values of H_k^n is by the formula

$$\sum_{k=1}^{n}H_k^n=\mu(n)\quad\text{for all } n\ge 1, \tag{20}$$

which, in a sense, gives a new representation of the Möbius function. The proof is very easy. In expression (11) of Theorem 2, just choose $S(n)=1$ for all $n \geq 1$. This makes $A(n)=\mu(n)$ for all $n \geq 0$. But then, by relation (17), we have result (20) immediately.

A Table of the Numbers H_k^n for $0 \leq k \leq n$, $n = 0(1)18$

n																			
0	1																		
1	−1	1																	
2	1	−2	1																
3	1	−1	−1	1															
4	0	1	−1	−1	1														
5	1	−1	0	0	−1	1													
6	−1	2	0	−1	0	−1	1												
7	1	−1	0	0	0	0	−1	1											
8	0	0	0	1	−1	0	0	−1	1										
9	0	0	1	−1	0	0	0	0	−1	1									
10	−1	2	−1	0	1	−1	0	0	0	−1	1								
11	1	−1	0	0	0	0	0	0	0	0	−1	1							
12	0	−1	1	1	−1	1	−1	0	0	0	0	−1	1						
13	1	−1	0	0	0	0	0	0	0	0	0	0	−1	1					
14	−1	2	−1	0	0	0	1	−1	0	0	0	0	0	−1	1				
15	−1	1	1	−1	1	−1	0	0	0	0	0	0	0	0	−1	1			
16	0	0	0	0	0	0	0	1	−1	0	0	0	0	0	0	−1	1		
17	1	−1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	1	
18	0	0	−1	1	0	1	−1	0	1	−1	0	0	0	0	0	0	0	−1	1
$k =$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18

If we adopt the convention that $H_0^n = -\mu(n)$, but with $H_0^0 = 1$, then (20) may be reformulated to say that

$$\sum_{k=0}^{n} H_k^n = 0, \quad \text{for all } n \geq 1. \tag{21}$$

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